

**Models.** • In  $[x, y] = \delta x$ ,  $xf(y) = f(y + \delta)x$ . If  $\delta^2 = 0$ ,  $[x, f(y)] = \delta f'(y)x$ .

• In  $[x, y] = \delta x + z^2$ ,  $xf(y) = f(y + \delta)x + \frac{z^2}{\delta}(f(y + \delta) - f(y))$ . If  $\delta^2 = 0$ ,  $[x, f(y)] = \delta f'(y)x + z^2 f'(y)$ .

• If  $S_n := \sum_{k=0}^{n-1} A^k C B^{n-1-k}$  then  $AS_n - S_n B = A^n C - C B^n$  so  $S_n = (L_A - R_B)^{-1}(A^n C - C B^n)$ .

• If  $\psi(x) = \sum_{n \geq 0} a_n x^n$  then  $\sum_{n \geq 0} a_n \sum_{k=0}^{n-1} b^n (-b)^{n-1-k} = (\psi(b) - \psi(-b))/2b$ .

**Deriving Gassner.**  $\mathcal{L}^{2Dw}$  is  $\mathbb{Q}[[b_i]]\langle a_{ij} \rangle$  modulo locality,  $[a_{ij}, a_{ik}] = 0$ ,  $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$ , and  $[a_{ij}, a_{ji}] = b_i a_{ji} - b_j a_{ij}$ . Acts on  $\mathbf{V} = \mathbb{Q}[[b_i]]\langle x_i = a_{i\infty} \rangle$  by  $[a_{ij}, x_i] = 0$ ,  $[a_{ij}, x_j] = b_i x_j - b_j x_i$ . Hence  $e^{\text{ad}_{a_{ij}}} x_i = x_i$ ,  $e^{\text{ad}_{a_{ij}}} x_j = e^{b_i} x_j + \frac{b_i}{b_i}(1 - e^{b_i})x_i$ . Renaming  $y_i = x_i/b_i$ ,  $t_i = e^{b_i}$ ,

$$\text{get } [e^{\text{ad}_{a_{ij}}}]_{y_i, y_j} = \begin{pmatrix} 1 & 1 - t_i \\ 0 & t_i \end{pmatrix}.$$

**The  $\mathcal{L}^{2Dw}$  Adjoint representation.**  $e^{\text{ad}_{a_{ij}}}$  acts by

$$a_{kl} \mapsto a_{kl}, \quad a_{ik} \mapsto a_{ik}, \quad a_{kj} \mapsto e^{-b_i} a_{kj} + \frac{b_k}{b_i}(1 - e^{-b_i})a_{ij},$$

$$a_{ki} \mapsto a_{ki} + (1 - e^{-b_i})a_{kj} + b_k \frac{e^{-b_i} - 1}{b_i} a_{ij},$$

$$a_{jk} \mapsto e^{b_i} a_{jk} + \frac{b_j}{b_i}(1 - e^{b_i})a_{ik}, \quad a_{ji} \mapsto e^{b_i} a_{ji} + \frac{b_j}{b_i}(1 - e^{b_i})a_{ij}.$$

**Adjoint Gassner.** Renaming  $\alpha_{ij} = a_{ij}/b_i$  and  $t_i = e^{b_i}$ , get

$$\alpha_{kj} \mapsto t_i^{-1} \alpha_{kj} + (1 - t_i^{-1}) \alpha_{ij},$$

$$\alpha_{ki} \mapsto \alpha_{ki} + (1 - t_i^{-1}) \alpha_{kj} + (t_i^{-1} - 1) \alpha_{ij}$$

$$\alpha_{jk} \mapsto t_i \alpha_{jk} + (1 - t_i) \alpha_{ik}, \quad \alpha_{ji} \mapsto t_i \alpha_{ji} + (1 - t_i) \alpha_{ij}.$$

Implementation/verification: [pensieve://2015-04/nb/ZeroCo.pdf](http://pensieve://2015-04/nb/ZeroCo.pdf). Interpretation:  $\pi_T$ -Artin?

**2Dv.**  $b$ : bracket trace;  $c$ : cobracket trace;  $\langle b, c \rangle = \delta \in \{0, 1\}$ ;  $\deg b_i = \deg c_j = \deg a_{ij} = \deg \delta = 1$ .

$\mathcal{A}^{2Dv}$  is  $\mathbb{Q}[[\delta]]FA(b_i, c_j, a_{ij})$  (so  $\mathcal{L}^v = \{f + f^{ij} a_{ij}\}$ ) modulo locality,

**tt.**  $[a_{ij}, a_{ik}] = c_k a_{ij} - c_j a_{ik} =: \gamma_{ijk}$ , (note  $\gamma_{i(jk)} = 0$ )

**hh.**  $[a_{ik}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$ ,

**ht.**  $[a_{ij}, a_{jk}] = b_i a_{jk} - b_j a_{ik} - \gamma_{ijk}$ ,

$\leq$ ,  $[a_{ij}, a_{ji}] = ?$ ,

**ab, ac.**  $[a_{ij}, b_i] = -[a_{ij}, b_j] = -[a_{ij}, c_i] = [a_{ij}, c_j]$

$$= \delta a_{ij} - b_i c_j =: \gamma_{ij},$$

**bc.**  $[b_i, c_j] = 0$ .

So  $a_{ij} f = f^\delta a_{ij} - \frac{b_i c_j}{\delta} (f^\delta - f)$ ,  $[a_{ij}, f] = (f^\delta - f) \left( a_{ij} - \frac{b_i c_j}{\delta} \right)$ ,

with  $f^\delta := f \left\| \begin{matrix} b_i \rightarrow b_i + \delta & b_j \rightarrow b_j - \delta \\ c_i \rightarrow c_i - \delta & c_j \rightarrow c_j + \delta \end{matrix} \right.$ .

**The Ascending Algebra  $\mathcal{A}_+^{2Dv}$ .** Same but with only  $a_{ij}$ ,  $i < j$ .

**The primitivity condition.**  $\ker(f + f^{ij} a_{ij} \mapsto \delta f + f^{ij} b_i c_j)$ . (Ignoring multiple arrows).

**The OneCo Quotient** is  $\delta^2 = \delta c_i = c_j c_k = 0$ , so

$$\mathcal{L}^{1co} = \left\{ (f + f^k c_k) + (f^{ij} + f^{ijk} c_k) a_{ij} + \delta f^{ijkl} a_{ij} a_{kl} : \right.$$

$$\left. f, f^{ij} \in \mathbb{Q}[[\delta, b_i]]; f^i, f^{ijk}, f^{ijkl} \in \mathbb{Q}[[b_i]] \right\}.$$

Then  $[a_{ij}, f + f^k c_k] = (\partial_i f - \partial_j f - f^i + f^j) \gamma_{ij}$  and

**$\gamma b$ .**  $[\gamma_{ijk}, b_l] = 0$  incl.  $l \in \{i, j, k\}$ ,

**$\gamma tt$ .**  $[\gamma_{ijk}, a_{il}] = 0$ ,

**$\gamma ht$ .**  $[\gamma_{ijk}, a_{jl}] = b_j \gamma_{ikl} - \gamma_{ik} a_{jl}$ ,

**$\gamma th$ .**  $[\gamma_{ijk}, a_{li}] = b_i \gamma_{ljk} - b_l \gamma_{ijk}$ ,

**$\gamma hh$ .**  $[\gamma_{ijk}, a_{lj}] = b_l \gamma_{ijk} + \gamma_{ik} a_{lj}$ ,

$$[\gamma_{ijk}, a_{ij}] = \gamma_{ij} a_{ik},$$

$$[\gamma_{ijk}, a_{jk}] = b_j \gamma_{ijk} - \gamma_{ij} a_{jk} - \gamma_{ik} a_{jk},$$

$$[\gamma_{ij}, a_{ik}] = 0,$$

$$[\gamma_{ik}, a_{jk}] = b_j \gamma_{ik},$$

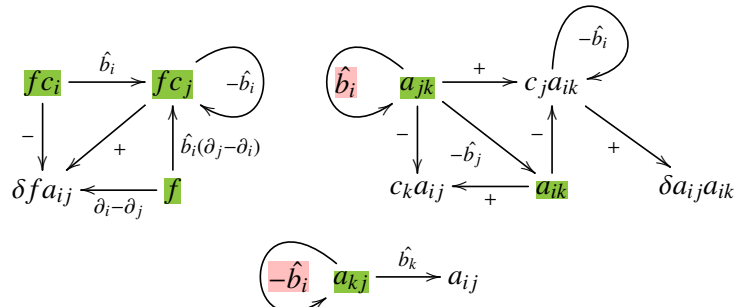
$$[\gamma_{ij}, a_{jk}] = -b_j \gamma_{ik},$$

$$[\gamma_{jk}, a_{ij}] = b_j \gamma_{ik} - b_i \gamma_{jk}.$$

(Is there a residual 4T?)

**State Diagrams.**  $\text{ad } a_{ij}$  yields

**green:** roots. **pink:** wrong.



so with  $\phi_0 := \phi(0)$ ,  $\phi_1 := \phi'_0$ , and  $\phi_1(x) := (\phi(x) - \phi_0)/x$ ,  $\phi(\text{ad } a_{ij})$  is

$$f c_i \mapsto \phi_0 f c_i + (b_i \phi_{\downarrow}(-b_i) - \phi_1) \delta f a_{ij} + b_i \phi_{\downarrow}(-b_i) f c_j$$

$$f c_j \mapsto \phi(-b_i) f c_j + \phi_{\downarrow}(-b_i) \delta f a_{ij}$$

$$f \mapsto \phi_0 f + b_i \phi_{\downarrow}(-b_i) (\partial_j f - \partial_i f) c_j + (b_i \phi_{\downarrow}(-b_i) - \phi_1) (\partial_j f - \partial_i f) \delta a_{ij}$$

$\delta a_{..} \mapsto$  as in Adjoint Gassner

$$a_{ik} \mapsto \phi_0 a_{ik} + \phi_1 c_k a_{ij} - \phi_{\downarrow}(-b_i) c_j a_{ik} - \phi_{\downarrow}(-b_i) \delta a_{ij} a_{ik}$$

$$a_{jk} \mapsto \phi(b_i) a_{jk} - (\phi_{\downarrow}(b_i) + b_j \phi_{\downarrow}(b_i)) c_k a_{ij} - b_j \phi_{\downarrow}(b_i) a_{ik} + \frac{\phi(b_i) - \phi(-b_i) + b_j (\phi_{\downarrow}(b_i) - \phi_{\downarrow}(-b_i))}{2b_i} c_j a_{ik} + \frac{\phi_{\downarrow}(b_i) - \phi_{\downarrow}(-b_i) + b_j (\phi_{\downarrow}(b_i) - \phi_{\downarrow}(-b_i))}{2b_i} \delta a_{ij} a_{ik}$$

$$a_{kj} \mapsto$$

$$a_{ij} \mapsto a_{ij}$$

**To do.** • Perhaps I should find a way to highlight the fact that  $v$  is a perturbation of  $w$ . • Position  $\text{FiC}$ . • Position the 2D Lie bialgebras.