The Adjoint representation of Aw/2D

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(150412) Deriving Gassner: $[a_{ij}, a_{ik}] = 0$, $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$, $[a_{ij}, a_{ji}] = b_i a_{ji} - b_j a_{ij}$, b_i central. Acts on $V = \mathbb{Q}[\![b_i]\!]\langle x_i = a_{i\infty}\rangle$ by $[a_{ij}, x_i] = 0$, $[a_{ij}, x_j] = b_i x_j - b_j x_i$. Hence $e^{\operatorname{ad} a_{ij}} x_i = x_i$, $e^{\operatorname{ad} a_{ij}} x_j = e^{b_i} x_j + \frac{b_j}{b_i}(1 - e^{b_i}) x_i$. Renaming $y_i = x_i/b_i$, $t_i = e^{b_i}$, get $[e^{\operatorname{ad} a_{ij}}]_{y_i, y_j} = \begin{pmatrix} 1 & 1-t_i \\ 0 & t_i \end{pmatrix}$.

Results for $e^{\operatorname{ad} a_{ij}}$:

$a_{kl} \longmapsto a_{kl}$ $\quad a_{ik} \longmapsto a_{ik}$

$a_{jk} \longmapsto e^{b_i} a_{jk} + \frac{b_j}{b_i}(1 - e^{b_i}) a_{ik}$ $\quad$ (also works for $k=i$)

$a_{kj} \longmapsto e^{-b_i} a_{kj} + \frac{b_k}{b_i}(1 - e^{-b_i}) a_{ij}$

$a_{ji} \longmapsto e^{b_j} a_{ji} + \frac{b_i}{b_j}(1 - e^{b_j}) a_{ij}$

Calculations:

$a_{kj} \longmapsto b_k a_{ij} - b_i a_{kj}$ $\quad [a_{ij}]_{a_{ij}, a_{kj}} = \begin{pmatrix} 0 & b_k \\ 0 & -b_i \end{pmatrix}$

$$\exp\begin{pmatrix} 0 & b_k \\ 0 & -b_i \end{pmatrix} = \begin{pmatrix} 1 & b_k \frac{e^{-b_i}-1}{-b_i} \\ 0 & e^{-b_i} \end{pmatrix}$$

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In[4]:= MatrixExp[( 0  bk
                    0 -bi )] // Simplify // MatrixForm
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Out[4]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{(1-e^{-b_i})\, b_k}{b_i} \\ 0 & e^{-b_i} \end{pmatrix}$$

$a_{ji} \longrightarrow b_i a_{ji} - b_j a_{ij}$ $\quad [a_{ij}]_{a_{ij}, a_{ji}} = \begin{pmatrix} 0 & -b_j \\ 0 & b_i \end{pmatrix}$

$\xrightarrow{\exp} \Big(1 \quad -b_j \frac{e^{b_i}-1}{b_i}$

$$\xrightarrow{exp} \begin{pmatrix} 1 & -b_j\frac{e^{b_i}-1}{b_i} \\ 0 & e^{b_i} \end{pmatrix}$$