

OneCo-150428

April-27-15 10:34 AM

model: derivation  $D$  w/  $Dy = by$ ,  $Db = c$ ,  $[b, c] = 0$ ,  $c^2 = 0$   
 what's  $e^{tD}y$ ?  $\det D = \deg b = 1$ ,  $\deg c = 2$ .

$$\text{Ans: } e^{tD}y = e^{tb}y +$$

Wrong Example:  $D = \partial_x$ ,  $y = e^{x^2/2}$

$$Dy = x e^{x^2/2} =: by$$

$$Db = 1 \quad D1 = 0$$

$$\begin{aligned} e^{tD}y &= e^{(x+t)^2/2} = e^{tx+t^2}y \\ &= e^{tb+t^2c}y \end{aligned}$$

# Cheat Sheet OneCo

http://drorbn.net/AcademicPensieve/2015-04/  
initiated 14/4/15, modified 28/4/15, 6:46am

**Background.**  $\delta e^\gamma = e^\gamma \cdot \left( \delta\gamma \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma} \right) = \left( \delta\gamma \frac{e^{\text{ad } \gamma} - 1}{\text{ad } \gamma} \right) \cdot e^\gamma$

The differential of  $\gamma = \text{bch}(\alpha, \beta)$ :

$$\delta\gamma \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma} = \left( \delta\alpha \frac{1 - e^{-\text{ad } \alpha}}{\text{ad } \alpha} \right) \frac{1 - e^{-\text{ad } \beta}}{\text{ad } \beta} + \left( \delta\beta \frac{1 - e^{-\text{ad } \beta}}{\text{ad } \beta} \right)$$

**Models.** • In  $[x, y] = \delta x$ ,  $x f(y) = f(y + \delta)x$ . If  $\delta^2 = 0$ ,  $[x, f(y)] = \delta f'(y)x$ .

• In  $[x, y] = \delta x + z^2$ ,  $x f(y) = f(y + \delta)x + \frac{z^2}{\delta}(f(y + \delta) - f(y))$ . If  $\delta^2 = 0$ ,  $[x, f(y)] = \delta f'(y)x + z^2 f''(y)$ .

• If  $S_n := \sum_{k=0}^{n-1} A^k C B^{n-1-k}$  then  $A S_n - S_n B = A^n C - C B^n$  so  $S_n = (L_a - R_B)^{-1}(A^n C - C B^n)$ .

• If  $\psi(x) = \sum_{n \geq 0} a_n x^n$  then  $\sum_{n \geq 0} a_n \sum_{k=0}^{n-1} b^n (-b)^{n-1-k} = (\psi(b) - \psi(-b))/2b$ .

**Deriving Gassner.**  $\mathcal{L}^{2Dw}$  is  $\mathbb{Q}\langle b_i \rangle \langle a_{ij} \rangle$  modulo locality,  $[a_{ij}, a_{ik}] = 0$ ,  $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$ , and  $[a_{ij}, a_{ji}] = b_i a_{ji} - b_j a_{ij}$ . Acts on  $V = \mathbb{Q}\langle b_i \rangle \langle x_i = a_{i\infty} \rangle$  by  $[a_{ij}, x_i] = 0$ ,  $[a_{ij}, x_j] = b_i x_j - b_j x_i$ . Hence  $e^{\text{ad } a_{ij}} x_i = x_i$ ,  $e^{\text{ad } a_{ij}} x_j = e^{b_i} x_j + \frac{b_j}{b_i}(1 - e^{b_i})x_i$ . Renaming  $y_i = x_i/b_i$ ,  $t_i = e^{b_i}$ , get  $[e^{\text{ad } a_{ij}}]_{y_i, y_j} = \begin{pmatrix} 1 & 1 - t_i \\ 0 & t_i \end{pmatrix}$ .

**The  $\mathcal{L}^{2Dw}$  Adjoint representation.**  $e^{\text{ad } a_{ij}}$  acts by

$$a_{kl} \mapsto a_{kl}, \quad a_{ik} \mapsto a_{ik}, \quad a_{kj} \mapsto e^{-b_i} a_{kj} + \frac{b_k}{b_i}(1 - e^{-b_i}) a_{ij},$$

$$a_{ki} \mapsto a_{ki} + (1 - e^{-b_i}) a_{kj} + b_k \frac{e^{-b_i} - 1}{b_i} a_{ij}$$

$$a_{jk} \mapsto e^{b_i} a_{jk} + \frac{b_j}{b_i}(1 - e^{b_i}) a_{ik}, \quad a_{ji} \mapsto e^{b_i} a_{ji} + \frac{b_j}{b_i}(1 - e^{b_i}) a_{ij}.$$

**Adjoint Gassner.** Renaming  $\alpha_{ij} = a_{ij}/b_i$  and  $t_i = e^{b_i}$ , get

$$\alpha_{kj} \mapsto t_i^{-1} \alpha_{kj} + (1 - t_i^{-1}) \alpha_{ij},$$

$$\alpha_{ki} \mapsto \alpha_{ki} + (1 - t_i^{-1}) \alpha_{kj} + (t_i^{-1} - 1) \alpha_{ij}$$

$$\alpha_{jk} \mapsto t_i \alpha_{jk} + (1 - t_i) \alpha_{ik}, \quad \alpha_{ji} \mapsto t_i \alpha_{ji} + (1 - t_i) \alpha_{ij}.$$

Implementation/verification: [pensieve://2015-04/nb/ZeroCo.pdf](http://pensieve://2015-04/nb/ZeroCo.pdf). Interpretation:  $\pi_T$ -Artin?

**2Dv.**  $b$ : bracket trace;  $c$ : cobracket trace;  $\langle b, c \rangle = \delta \in \{0, 1\}$ ;  $\text{deg } b_i = \text{deg } c_j = \text{deg } a_{ij} = \text{deg } \delta = 1$ .

$\mathcal{A}^{2Dv}$  is  $\mathbb{Q}\langle \delta \rangle \langle FA(b_i, c_j, a_{ij}) \rangle$  (so  $\mathcal{L}^v = \{f + f^{ij} a_{ij}\}$ ) modulo locality,  $[a_{ij}, a_{ik}] = c_k a_{ij} - c_j a_{ik}$ ,  $[a_{ik}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$ ,  $[a_{ij}, a_{jk}] = (c_j - b_j) a_{ik} + b_i a_{jk} - c_k a_{ij}$ ,  $[a_{ij}, a_{ji}] = ?$ ,  $[a_{ij}, b_i] = -[a_{ij}, b_j] = -[a_{ij}, c_i] = [a_{ij}, c_j] = \delta a_{ij} - b_i c_j$ ,  $[b_i, c_j] = 0$ .

$$a_{ij} f = f^\delta a_{ij} - \frac{b_i c_j}{\delta} (f^\delta - f), \quad [a_{ij}, f] = (f^\delta - f) \left( a_{ij} - \frac{b_i c_j}{\delta} \right),$$

with  $f^\delta := f \frac{1 - e^{-\text{ad } \delta}}{\text{ad } \delta}$ .

**The Ascending Algebra  $\mathcal{A}_+^{2Dv}$ .** Same but with only  $a_{ij}$ ,  $i < j$ .

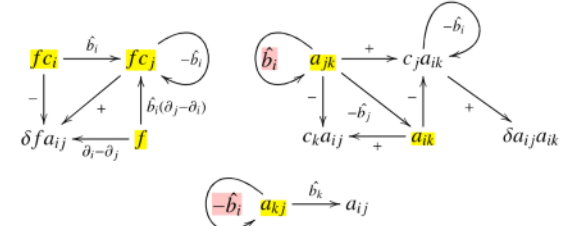
**The primitivity condition.**  $\ker(f + f^{ij} a_{ij} \mapsto \delta f + f^{ij} b_i c_j)$ . (Ignoring multiple arrows).

**The OneCo Quotient** is  $\delta^2 = \delta c_i = c_j c_k = 0$ , so

$$\mathcal{L}^{1co} = \left\{ (f + f^k c_k) + (f^{ij} + f^{ijk} c_k) a_{ij} + \delta f^{ijkl} a_{ij} a_{kl} : f, f^{ij} \in \mathbb{Q}\langle \delta, b_i \rangle; f^i, f^{ijk}, f^{ijkl} \in \mathbb{Q}\langle b_i \rangle \right\}.$$

Then  $[a_{ij}, f + f^k c_k] = (\partial_i f - \partial_j f - f^i + f^j)(\delta a_{ij} - b_i c_j)$ .

**State Diagrams.**  $\text{ad } a_{ij}$  yields **yellow**: roots, **pink**: wrong.



so with  $\phi_0 := \phi(0)$ ,  $\phi_1 := \phi'_0$ , and  $\phi_{\perp}(x) := (\phi(x) - \phi_0)/x$ ,  $\phi(\text{ad } a_{ij})$  is

$$f c_i \mapsto \phi_0 f c_i + (b_i \phi_{\perp}(-b_i) - \phi_1) \delta f a_{ij} + b_i \phi_{\perp}(-b_i) f c_j$$

$$f c_j \mapsto \phi(-b_i) f c_j + \phi_{\perp}(-b_i) \delta f a_{ij}$$

$$f \mapsto \phi_0 f + b_i \phi_{\perp}(-b_i) (\partial_j f - \partial_i f) c_j + (b_i \phi_{\perp}(-b_i) - \phi_1) (\partial_j f - \partial_i f) \delta a_{ij}$$

$\delta a_{..}$   $\mapsto$  as in Adjoint Gassner

$$a_{ik} \mapsto \phi_0 a_{ik} + \phi_1 c_k a_{ij} - \phi_{\perp}(-b_i) c_j a_{ik} - \phi_{\perp}(-b_i) \delta a_{ij} a_{ik}$$

$$a_{jk} \mapsto \phi(b_i) a_{jk} - (\phi_{\perp}(b_i) + b_j \phi_{\perp}(b_i)) c_k a_{ij} - b_j \phi_{\perp}(b_i) a_{ik} + \frac{\phi(b_i) - \phi(-b_i) + b_j (\phi_{\perp}(b_i) - \phi_{\perp}(-b_i))}{2b_i} c_j a_{ik}$$

$$+ \frac{\phi_{\perp}(b_i) - \phi_{\perp}(-b_i) + b_j (\phi_{\perp}(b_i) - \phi_{\perp}(-b_i))}{2b_i} \delta a_{ij} a_{ik}$$

$$a_{kj} \mapsto$$

$$a_{ij} \mapsto a_{ij}$$

**To do.** • Perhaps I should find a way to highlight the fact that  $v$  is a perturbation of  $w$ . • Position Fic. • Position the 2D Lie bialgebras.

From 2011-08 / cliff & Halachova:

Duf. A Lie bialgebra structure on a Lie algebra  $\mathfrak{g}$ .

Examples. 1.  $\delta_{\mathfrak{g}} = 0$

2. 2D:  $[x, y] = x$ , after rescaling, with  $\delta x = \alpha(\delta x \wedge y)$   $\alpha = 0 = \beta$  or

$$\left. \begin{array}{l} \alpha, \beta \in \mathbb{R} \\ \delta x = \alpha(x^{\wedge}y) \\ \delta y = \beta(x^{\wedge}y) \end{array} \right\} \begin{array}{l} \text{with } \alpha, \beta \in \mathbb{R}, \text{ either} \\ \alpha = 0 = \beta \text{ or} \\ \alpha = 1, \beta = 0 \text{ or } \alpha = 0, \beta = 1. \end{array}$$

Neither of the two non-co-commutative versions  
seems to satisfy FIC!