Pensieve header: One-Co computations; continued pensieve://2015-05/.

The Adjoint action

```
e_p_[x_] := x^-p (e^x - Sum[x^k/k!, {k, 0, p-1}]);
Unprotect[NonCommutativeMultiply];
0 ** _ := 0; _ ** 0 := 0;
x_ ** (y_ + z_) := x ** y + x ** z;
(x_ + y_) ** z_ := x ** z + y ** z;
L[g_] := L[1, g];
L /: L[f1_, g_] + L[f2_, g_] := L[f1 + f2, g];
L /: c_ * L[f_, g_] := L[c * f, g];
L /: L[f1_, 1] ** L[f2_, g_] := L[f1 f2, g];
L /: L[f1_, δ] ** L[f2_, a[i_, j_]] := L[f1 f2, δa[i, j]];
L /: L[f1_, c[k_]] ** L[f2_, a[i_, j_]] := L[f1 f2, ca[k, i, j]];
L /: L[f1_, δa[i_, j_]] ** L[f2_, a[k_, l_]] := L[f1 f2, δa[i, j, k, l]];
L /: L[_, _c | _ca | _δa] ** L[_, _c | _ca | _δa] := 0;
LSimp[L[f_, g_]] := L[FullSimplify[f], g] /. L[0, _] :> 0;
LSimp[expr_] := expr /. λ_L :> LSimp[λ];
expr_ // DegreeScale[t_] := expr /. L[f_, g_] :> LSimp[L[
      (f /. b_i_ :> t b_i) * Switch[g,
        1, 1,
        δ, t,
        c[_], t,
        a[_, _], t,
        ca[_, _, _], t^2,
        δa[_, _, _, _], t^3
       ], g
     ]];
expr_ // Ad[t_ * a[i_, j_]] :=
  expr // DegreeScale[t^-1] // Ad[a[i, j]] // DegreeScale[t];
DistinctQ[is___] := (Sort[{is}] === Union[{is}]);
expr_ // Ad[a[i_, j_]] := Expand[expr /. {
    L[f_, c@i] :> L[f, c@i] + L[(e^-b_i - 1)/b_i f, δa[i, j]] + L[(1 - e^-b_i) f, c@j],
    L[f_, c@j] :> L[e^-b_i f, c@j] + L[(1 - e^-b_i)/b_i f, δa[i, j]],
    L[f_, c@k_] /; k ≠ i && k ≠ j :> L[f, c@k],
    L[f_, δ] :> L[f, δ],
    L[f_, 1] :>
```

```
  L[f, 1] + L[(1 - e^(-b_i)) (∂_(b_j) f - ∂_(b_i) f), c@j] + L[(e^(-b_i) - 1)/b_i (∂_(b_j) f - ∂_(b_i) f), δa[i, j]],
 L[f_, δa[k_, j]] /; DistinctQ[i, j, k] :>
  L[e^(-b_i) f, δa[k, j]] + L[b_k/b_i (1 - e^(-b_i)) f, δa[i, j]],
 L[1, a[j, k_]] /; DistinctQ[i, j, k] :> L[e^(b_i), a[j, k]] -
    L[(b_j + b_i (1 + b_j) - e^(b_i) (b_i + b_j))/b_i^2, ca[k, i, j]] + L[b_j/b_i (1 - e^(b_i)), a[i, k]] +
    L[(Sinh[b_i] b_i + (-1 + Cosh[b_i]) b_j)/b_i^2, ca[j, i, k]] +
    L[(b_i (-1 + Cosh[b_i] - b_j) + Sinh[b_i] b_j)/b_i^3, δa[i, j, i, k]],
 L[f_, δa[j, k_]] /; DistinctQ[i, j, k] :>
  L[e^(b_i) f, δa[j, k]] + L[b_j/b_i (1 - e^(b_i)) f, δa[i, k]],
 L[f_, δa[k_, i]] /; DistinctQ[i, j, k] :>
  L[f, δa[k, i]] + L[(1 - e^(-b_i)) f, δa[k, j]] + L[b_k/b_i (e^(-b_i) - 1) f, δa[i, j]],
 L[f_, δa[j, i]] :> L[e^(b_i) f, δa[j, i]] + L[b_j/b_i (1 - e^(b_i)) f, δa[i, j]],
 L[1, a[i, k_]] :> L[1, a[i, k]] + L[1, ca[k, i, j]] +
    L[(e^(-b_i) - 1)/b_i, ca[j, i, k]] + L[(1 - b_i - e^(-b_i))/b_i^2, δa[i, j, i, k]],
 L[f_, δa[i, k_]] :> L[f, δa[i, k]],
 L[f_, a[k_, l_]] /; DistinctQ[i, j, k, l] :> L[f, a[k, l]],
 L[f_, δa[k_, l_]] /; DistinctQ[i, j, k, l] :> L[f, δa[k, l]],
 L[f_, ca[k_, l_, m_]] :>
  (L[f, c[k]] // Ad[a[i, j]]) ** ((L[1, δa[l, m]] // Ad[a[i, j]]) /. δa → a),
 L[f_, δa[k_, l_, m_, n_]] :> (L[f, δa[k, l]] // Ad[a[i, j]]) **
   ((L[1, δa[m, n]] // Ad[a[i, j]]) /. δa → a),
 L[f_, g_] /; (f =!= 1) && (g =!= 1) :>
  (L[f, 1] // Ad[a[i, j]]) ** (L[1, g] // Ad[a[i, j]]),
 L[f_, g_] → TBD[a[i, j], f, g]
}]
```

```
-(e1[b_i] + b_j e2[b_i]) // FullSimplify
```

$$\frac{b_j + b_i\,(1 + b_j) - e^{b_i}\,(b_i + b_j)}{b_i^2}$$

```
(e0[b_i] - e0[-b_i] + b_j (e1[b_i] - e1[-b_i]))/(2 b_i) // FullSimplify
```

$$\frac{\text{Sinh}[b_i]\,b_i + (-1 + \text{Cosh}[b_i])\,b_j}{b_i^2}$$

```
(e1[bi] - e1[-bi] + bj (e2[bi] - e2[-bi]))/(2 bi) // FullSimplify
```

$$\frac{b_i\,(-1+\text{Cosh}[b_i]-b_j)+\text{Sinh}[b_i]\,b_j}{b_i^3}$$

```
Series[(bi (-1 + Cosh[bi] - bj) + Sinh[bi] bj)/bi^3, {bi, 0, 3}]
```

$$\frac{1}{6}\,(3+b_j)+\frac{1}{120}\,(5+b_j)\,b_i^2+O[b_i]^4$$

The semi-group property

```
L[1, a[1, 4]] // Ad[x a[1, 2]] // Ad[y a[1, 2]]
```

$$L[1,a[1,4]]+L[x+y,ca[4,1,2]]+L\Big[\frac{-1+e^{-(x+y)\,b_1}}{b_1},ca[2,1,4]\Big]+L\Big[-\frac{-1+e^{-(x+y)\,b_1}+(x+y)\,b_1}{b_1^2},\delta a[1,2,1,4]\Big]$$

```
L[1, a[1, 4]] // Ad[(x + y) a[1, 2]]
```

$$L[1,a[1,4]]+L[x+y,ca[4,1,2]]+L\Big[\frac{-1+e^{-(x+y)\,b_1}}{b_1},ca[2,1,4]\Big]+L\Big[-\frac{-1+e^{-(x+y)\,b_1}+(x+y)\,b_1}{b_1^2},\delta a[1,2,1,4]\Big]$$

```
L[1, a[2, 4]] // Ad[x a[1, 2]]
```

$$L\Big[e^{x\,b_1},a[2,4]\Big]+L\Big[-\frac{\left(-1+e^{x\,b_1}\right)b_2}{b_1},a[1,4]\Big]+L\Big[\frac{\text{Sinh}[x\,b_1]\,b_1+(-1+\text{Cosh}[x\,b_1])\,b_2}{b_1^2},ca[2,1,4]\Big]+L\Big[\frac{\text{Sinh}[x\,b_1]\,b_2+b_1\,(-1+\text{Cosh}[x\,b_1]-x\,b_2)}{b_1^3},\delta a[1,2,1,4]\Big]+L\Big[-\frac{b_2-e^{x\,b_1}\,(b_1+b_2)+b_1\,(1+x\,b_2)}{b_1^2},ca[4,1,2]\Big]$$

t1 = L[1, a[2, 4]] // Ad[x a[1, 2]] // Ad[y a[1, 2]]

$$L\left[e^{(x+y)\,b_1},\ a[2,\ 4]\right]+L\left[-e^{x\,b_1}\left(-1+e^{y\,b_1}\right)x,\ ca[2,\ 2,\ 4]\right]+$$
$$L\left[\frac{e^{x\,b_1}\left(-1+e^{y\,b_1}\right)x}{b_1},\ \delta a[1,\ 2,\ 2,\ 4]\right]+L\left[-\frac{\left(-1+e^{(x+y)\,b_1}\right)b_2}{b_1},\ a[1,\ 4]\right]+$$
$$L\left[\frac{1}{b_1^2}e^{-y\,b_1}\left(\left(-1+\text{Cosh}[x\,b_1]+e^{(x+y)\,b_1}\left(-1+\text{Cosh}[y\,b_1]\right)\right)b_2+\right.\right.$$
$$\left.\left.b_1\left(-1+e^{y\,b_1}+\frac{1}{2}e^{x\,b_1}\left(-1+e^{y\,b_1}\right)^2+\text{Sinh}[x\,b_1]+e^{(x+y)\,b_1}\left(-1+e^{y\,b_1}\right)x\,b_2\right)\right),\ ca[2,\ 1,\ 4]\right]+$$
$$L\left[\frac{1}{2\,b_1^3}e^{-(x+y)\,b_1}\left(-b_2-e^{x\,b_1}\left(-e^{(x+2\,y)\,b_1}-2\left(-1+e^{x\,b_1}\right)\left(-1+e^{y\,b_1}\right)\right)b_2+b_1\left(1+2\,e^{x\,b_1}-2\,e^{2\,x\,b_1}-\right.\right.\right.$$
$$\left.\left.\left.4\,e^{(x+y)\,b_1}+e^{2\,(x+y)\,b_1}+2\,e^{(2\,x+y)\,b_1}-2\,e^{(x+y)\,b_1}\left(\left(1+e^{x\,b_1}\left(-1+e^{y\,b_1}\right)\right)x+y\right)b_2\right)\right),$$
$$\delta a[1,\ 2,\ 1,\ 4]\right]+L\left[-\frac{b_2-e^{(x+y)\,b_1}\left(b_1+b_2\right)+b_1\left(1+\left(x+\left(-1+2\,e^{x\,b_1}\right)y\right)b_2\right)}{b_1^2},\ ca[4,\ 1,\ 2]\right]$$

t2 = L[1, a[2, 4]] // Ad[(x + y) a[1, 2]]

$$L\left[e^{(x+y)\,b_1},\ a[2,\ 4]\right]+L\left[-\frac{\left(-1+e^{(x+y)\,b_1}\right)b_2}{b_1},\ a[1,\ 4]\right]+$$
$$L\left[\frac{\text{Sinh}[(x+y)\,b_1]\,b_1+(-1+\text{Cosh}[(x+y)\,b_1])\,b_2}{b_1^2},\ ca[2,\ 1,\ 4]\right]+$$
$$L\left[\frac{\text{Sinh}[(x+y)\,b_1]\,b_2+b_1\,(-1+\text{Cosh}[(x+y)\,b_1]-(x+y)\,b_2)}{b_1^3},\ \delta a[1,\ 2,\ 1,\ 4]\right]+$$
$$L\left[-\frac{b_2-e^{(x+y)\,b_1}\,(b_1+b_2)+b_1\,(1+(x+y)\,b_2)}{b_1^2},\ ca[4,\ 1,\ 2]\right]$$

t1 - t2

$$L[0,\ a[1,\ 4]]+L[0,\ a[2,\ 4]]+$$
$$L\left[-e^{x\,b_1}\left(-1+e^{y\,b_1}\right)x,\ ca[2,\ 2,\ 4]\right]+L\left[\frac{e^{x\,b_1}\left(-1+e^{y\,b_1}\right)x}{b_1},\ \delta a[1,\ 2,\ 2,\ 4]\right]+$$
$$L\left[-\frac{\text{Sinh}[(x+y)\,b_1]\,b_1+(-1+\text{Cosh}[(x+y)\,b_1])\,b_2}{b_1^2}+\frac{1}{b_1^2}\right.$$
$$e^{-y\,b_1}\left(\left(-1+\text{Cosh}[x\,b_1]+e^{(x+y)\,b_1}\left(-1+\text{Cosh}[y\,b_1]\right)\right)b_2+\right.$$
$$\left.\left.b_1\left(-1+e^{y\,b_1}+\frac{1}{2}e^{x\,b_1}\left(-1+e^{y\,b_1}\right)^2+\text{Sinh}[x\,b_1]+e^{(x+y)\,b_1}\left(-1+e^{y\,b_1}\right)x\,b_2\right)\right),$$
$$ca[2,\ 1,\ 4]\right]+L\left[-\frac{1}{b_1^3}(\text{Sinh}[(x+y)\,b_1]\,b_2+b_1\,(-1+\text{Cosh}[(x+y)\,b_1]-(x+y)\,b_2))+\right.$$
$$\frac{1}{2\,b_1^3}e^{-(x+y)\,b_1}\left(-b_2-e^{x\,b_1}\left(-e^{(x+2\,y)\,b_1}-2\left(-1+e^{x\,b_1}\right)\left(-1+e^{y\,b_1}\right)\right)b_2+b_1\left(1+2\,e^{x\,b_1}-2\,e^{2\,x\,b_1}-\right.\right.$$
$$\left.\left.4\,e^{(x+y)\,b_1}+e^{2\,(x+y)\,b_1}+2\,e^{(2\,x+y)\,b_1}-2\,e^{(x+y)\,b_1}\left(\left(1+e^{x\,b_1}\left(-1+e^{y\,b_1}\right)\right)x+y\right)b_2\right)\right),$$
$$\delta a[1,\ 2,\ 1,\ 4]\right]+L\left[\frac{b_2-e^{(x+y)\,b_1}\,(b_1+b_2)+b_1\,(1+(x+y)\,b_2)}{b_1^2}-\right.$$
$$\left.\frac{b_2-e^{(x+y)\,b_1}\,(b_1+b_2)+b_1\left(1+\left(x+\left(-1+2\,e^{x\,b_1}\right)y\right)b_2\right)}{b_1^2},\ ca[4,\ 1,\ 2]\right]$$

R

```
R[i_, j_] := Ad[a[i, j]]
```

Verifying R3

```
Verify[expr_] := Module[{lhs, rhs}, {
    lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // LSimp,
    rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // LSimp,
    LSimp[lhs - rhs] == 0
   }]
```

```
L[1, a[1, 4]] // Verify // ColumnForm
```

$$L[1, a[1, 4]] + L[1, ca[4, 1, 2]] + L[1, ca[4, 1, 3]] + L\left[\frac{-1+e^{-b_1}}{b_1}, ca[2, 1, 4]\right] + L\left[\frac{-1+e^{-b_1}}{b_1}, ca[3,$$

$$L[1, a[1, 4]] + L[1, ca[4, 1, 2]] + L[1, ca[4, 1, 3]] + L\left[\frac{-1+e^{-b_1}}{b_1}, ca[2, 1, 4]\right] + L\left[\frac{-1+e^{-b_1}}{b_1}, ca[3,$$

True

```
L[1, a[2, 4]] // Verify // ColumnForm
```

$$L\left[e^{b_1}, a[2, 4]\right] + L\left[\frac{e^{b_1}\,(1-e^{-b_2}-b_2)}{b_2^2}, \delta a[2, 3, 2, 4]\right] + L\left[\frac{e^{b_1}\,(-1+e^{-b_2})}{b_2}, ca[3, 2, 4]\right] + L\left[-\frac{(-1+e^{b_1})\,b_2}{b_1}, a[1\right.$$

$$L[1, ca[4, 2, 3]] + L\left[e^{b_1}, a[2, 4]\right] + L\left[\frac{e^{-b_1-b_2}\,(-1+e^{b_1})^2\,(-1+e^{b_2})}{b_1^2}, \delta a[1, 3, 1, 4]\right] + L\left[\frac{e^{-b_1}\,(1-e^{b_1})\,(-1+e^{-b_2})}{b_1}\right.$$

$$L\left[\frac{e^{-b_2}\,(-(-1+e^{b_1})\,(-1+e^{b_2})+(-e^{b_1}+e^{b_2})\,b_1)}{b_1^2}, \delta a[2, 3, 1, 4]\right] + L\left[-\frac{e^{-b_2}\,(-1+e^{b_1})\,(-1+e^{b_2})}{b_2}, ca[3, 2, 4]\right] + L\left[\frac{e^{-b_2}\,(-1+e^{-b_2})}{b_2}\right.$$

```
L[c@1] // Verify // ColumnForm
```

$$L[1, c[1]] + L\left[1-e^{-b_1}, c[2]\right] + L\left[1-e^{-b_1}, c[3]\right] + L\left[\frac{-1+e^{-b_1}}{b_1}, \delta a[1, 2]\right] + L\left[\frac{-1+e^{-b_1}}{b_1}, \delta a[1, 3]\right]$$

$$L[1, c[1]] + L\left[1-e^{-b_1}, c[2]\right] + L\left[1-e^{-b_1}, c[3]\right] + L\left[\frac{-1+e^{-b_1}}{b_1}, \delta a[1, 2]\right] + L\left[\frac{-1+e^{-b_1}}{b_1}, \delta a[1, 3]\right]$$

True

```
L[c@2] // Verify // ColumnForm
```

$$L\left[e^{-b_1}, c[2]\right] + L\left[e^{-b_1}\left(1-e^{-b_2}\right), c[3]\right] + L\left[\frac{1-e^{-b_1}}{b_1}, \delta a[1, 2]\right] + L\left[\frac{(1-e^{-b_1})\,(1-e^{-b_2})}{b_1}, \delta a[1, 3]\right] + L\left[\frac{-1+e^{-}}{b_2}\right.$$

$$L\left[e^{-b_1}, c[2]\right] + L\left[e^{-b_1}\left(1-e^{-b_2}\right), c[3]\right] + L\left[\frac{1-e^{-b_1}}{b_1}, \delta a[1, 2]\right] + L\left[\frac{e^{-b_1-b_2}\,(-1+e^{b_1})\,(-1+e^{b_2})}{b_1}, \delta a[1, 3]\right] + L$$

True

```
L[c@3] // Verify // ColumnForm
```

$$L\left[e^{-b_1-b_2}, c[3]\right] + L\left[\frac{e^{-b_2}\,(1-e^{-b_1})}{b_1}, \delta a[1, 3]\right] + L\left[\frac{1-e^{-b_2}}{b_2}, \delta a[2, 3]\right]$$

$$L\left[e^{-b_1-b_2}, c[3]\right] + L\left[\frac{e^{-b_1-b_2}\,(-1+e^{b_1})}{b_1}, \delta a[1, 3]\right] + L\left[\frac{1-e^{-b_2}}{b_2}, \delta a[2, 3]\right]$$

True

```
L[f[b1, b2, b3], 1] // Verify // ColumnForm
```

$$L[f[b_1, b_2, b_3], 1] + L\left[-\frac{e^{-b_2}\,(-1+e^{b_2})\,(f^{(0,0,1)}[b_1,b_2,b_3]-f^{(0,1,0)}[b_1,b_2,b_3])}{b_2}, \delta a[2, 3]\right] + L\left[\left(1-e^{-b_1}\right)\left(f^{(0,1,0}\right.\right.$$

$$L[f[b_1, b_2, b_3], 1] + L\left[\frac{(-1+e^{-b_2})\,(f^{(0,0,1)}[b_1,b_2,b_3]-f^{(0,1,0)}[b_1,b_2,b_3])}{b_2}, \delta a[2, 3]\right] + L\left[\left(1-e^{-b_1}\right)\left(f^{(0,1,0)}[b_1\right.\right.$$

True