Cheat Sheet OneCo

Background. $\delta e^\gamma = e^\gamma \cdot \left(\delta\gamma /\!\!/ \frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\right) = \left(\delta\gamma /\!\!/ \frac{e^{\operatorname{ad}\gamma}-1}{\operatorname{ad}\gamma}\right)\cdot e^\gamma$

The differential of $\gamma = \operatorname{bch}(\alpha,\beta)$:

$$\delta\gamma /\!\!/ \frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma} = \left(\delta\alpha /\!\!/ \frac{1-e^{-\operatorname{ad}\alpha}}{\operatorname{ad}\alpha} /\!\!/ e^{-\operatorname{ad}\beta}\right) + \left(\delta\beta /\!\!/ \frac{1-e^{-\operatorname{ad}\beta}}{\operatorname{ad}\beta}\right)$$

Models. • In $[x,y]=\delta x$, $xf(y)=f(y+\delta)x$. If $\delta^2=0$, $[x,f(y)]=\delta f'(y)x$.

• In $[x,y]=\delta x+z^2$, $xf(y)=f(y+\delta)x+\frac{z^2}{\delta}(f(y+\delta)-f(y))$. If $\delta^2=0$, $[x,f(y)]=\delta f'(y)x+z^2f'(y)$.

• If $S_n := \sum_{k=0}^{n-1}A^kCB^{n-1-k}$ then $AS_n - S_nB = A^nC-CB^n$ so $S_n=(L_a-R_B)^{-1}(A^nC-CB^n)$.

• If $\psi(x)=\sum_{n\geq 0}a_nx^n$ then $\sum_{n\geq 0}a_n\sum_{k=0}^{n-1}b^n(-b)^{n-1-k} = (\psi(b)-\psi(-b))/2b$.

Deriving Gassner. $\mathcal{L}^{2Dw}$ is $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo locality, $[a_{ij},a_{ik}]=0$, $[a_{ik},a_{jk}]=-[a_{ij},a_{jk}]=b_ja_{ik}-b_ia_{jk}$, and $[a_{ij},a_{ji}]=b_ia_{ji}-b_ja_{ij}$. Acts on $V=\mathbb{Q}[\![b_i]\!]\langle x_i=a_{i\infty}\rangle$ by $[a_{ij},x_i]=0$, $[a_{ij},x_j]=b_ix_j-b_jx_i$. Hence $e^{\operatorname{ad}a_{ij}}x_i=x_i$, $e^{\operatorname{ad}a_{ij}}x_j=e^{b_i}x_j+\frac{b_j}{b_i}(1-e^{b_i})x_i$. Renaming $y_i=x_i/b_i$, $t_i=e^{b_i}$, get $[e^{\operatorname{ad}a_{ij}}]_{y_i,y_j}=\begin{pmatrix}1 & 1-t_i\\ 0 & t_i\end{pmatrix}$.

The $\mathcal{L}^{2Dw}$ Adjoint representation. $e^{\operatorname{ad}a_{ij}}$ acts by

$$a_{kl}\mapsto a_{kl},\qquad a_{ik}\mapsto a_{ik},\qquad a_{kj}\mapsto e^{-b_i}a_{kj}+\frac{b_k}{b_i}(1-e^{-b_i})a_{ij},$$

$$a_{ki}\mapsto a_{ki}+(1-e^{-b_i})a_{kj}+b_k\frac{e^{-b_i}-1}{b_i}a_{ij}$$

$$a_{jk}\mapsto e^{b_i}a_{jk}+\frac{b_j}{b_i}(1-e^{b_i})a_{ik},\quad a_{ji}\mapsto e^{b_i}a_{ji}+\frac{b_j}{b_i}(1-e^{b_i})a_{ij}.$$

Adjoint Gassner. Renaming $\alpha_{ij}=a_{ij}/b_i$ and $t_i=e^{b_i}$, get

$$\alpha_{kj}\mapsto t_i^{-1}\alpha_{kj}+(1-t_i^{-1})\alpha_{ij},$$

$$\alpha_{ki}\mapsto a_{ki}+(1-t_i^{-1})\alpha_{kj}+(t_i^{-1}-1)\alpha_{ij}$$

$$\alpha_{jk}\mapsto t_i\alpha_{jk}+(1-t_i)\alpha_{ik},\quad \alpha_{ji}\mapsto t_i\alpha_{ji}+(1-t_i)\alpha_{ij}.$$

Implementation/verification: pensieve://2015-04/nb/ZeroCo.pdf.
Interpretation: π_T-Artin?

2Dv. b: bracket trace; c: cobracket trace; $\langle b,c\rangle=\delta\in\{0,1\}$; $\deg b_i=\deg c_j=\deg a_{ij}=\deg\delta=1$.
$\mathcal{A}^{2Dv}$ is $\mathbb{Q}[\![\delta]\!]FA(b_i,c_j,a_{ij})$ (so $\mathcal{L}^v=\{f+f^{ij}a_{ij}\}$) modulo locality, $[a_{ij},a_{ik}]=c_ka_{ij}-c_ja_{ik}$, $[a_{ik},a_{jk}]=b_ja_{ik}-b_ia_{jk}$, $[a_{ij},a_{jk}]=(c_j-b_j)a_{ik}+b_ia_{jk}-c_ka_{ij}$, $[a_{ij},a_{ji}]=?$, $[a_{ij},b_i]=-[a_{ij},b_j]=-[a_{ij},c_i]=[a_{ij},c_j]=\delta a_{ij}-b_ic_j$, $[b_i,c_j]=0$.

$$a_{ij}f=f^\delta a_{ij}-\frac{b_ic_j}{\delta}(f^\delta-f),\quad [a_{ij},f]=(f^\delta-f)\left(a_{ij}-\frac{b_ic_j}{\delta}\right),$$

with $f^\delta := f /\!\!/ \begin{pmatrix} b_i\to b_i+\delta & b_j\to b_j-\delta \\ c_i\to c_i-\delta & c_j\to c_j+\delta\end{pmatrix}$.

The Ascending Algebra $\mathcal{A}_+^{2Dv}$. Same but with only a_{ij}, $i<j$.

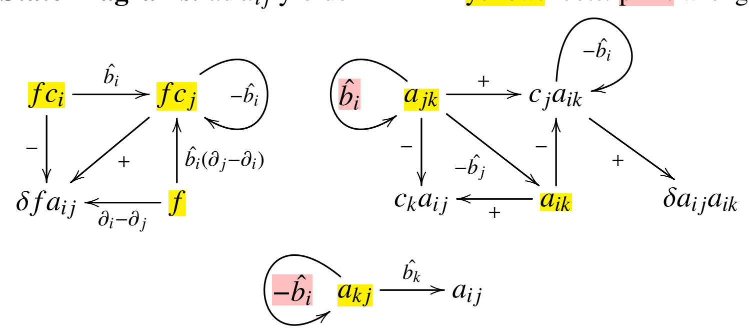
The primitivity condition. $\ker\left(f+f^{ij}a_{ij}\mapsto \delta f+f^{ij}b_ic_j\right)$. (Ignoring multiple arrows).

The OneCo Quotient is $\delta^2=\delta c_i=c_jc_k=0$, so

$$\mathcal{L}^{1co}=\Big\{(f+f^kc_k)+(f^{ij}+f^{ijk}c_k)a_{ij}+\delta f^{ijkl}a_{ij}a_{kl}:\\ f,f^{ij}\in\mathbb{Q}[\![\delta,b_i]\!];\ f^i,f^{ijk},f^{ijkl}\in\mathbb{Q}[\![b_i]\!]\Big\}.$$

Then $[a_{ij},f+f^kc_k]=(\partial_if-\partial_jf-f^i+f^j)(\delta a_{ij}-b_ic_j)$.

State Diagrams. $\operatorname{ad}a_{ij}$ yields yellow: roots. pink: wrong.



so with $\phi_0:=\phi(0)$, $\phi_1:=\phi_0'$, and $\phi_\downarrow(x):=(\phi(x)-\phi_0)/x$, $\phi(\operatorname{ad}a_{ij})$ is

$$fc_i\mapsto \phi_0fc_i+(b_i\phi_{\downarrow\downarrow}(-b_i)-\phi_1)\delta fa_{ij}+b_i\phi_\downarrow(-b_i)fc_j$$

$$fc_j\mapsto\phi(-b_i)fc_j+\phi_\downarrow(-b_i)\delta fa_{ij}$$

$$f\mapsto\phi_0f+b_i\phi_\downarrow(-b_i)(\partial_jf-\partial_if)c_j\\ +(b_i\phi_{\downarrow\downarrow}(-b_i)-\phi_1)(\partial_jf-\partial_if)\delta a_{ij}$$

$\delta a_{..}\mapsto$ as in Adjoint Gassner

$$a_{ik}\mapsto\phi_0a_{ik}+\phi_1c_ka_{ij}-\phi_\downarrow(-b_i)c_ja_{ik}-\phi_{\downarrow\downarrow}(-b_i)\delta a_{ij}a_{ik}$$

$$a_{jk}\mapsto\phi(b_i)a_{jk}-(\phi_\downarrow(b_i)+b_j\phi_{\downarrow\downarrow}(b_i))c_ka_{ij}-b_j\phi_\downarrow(b_i)a_{ik}\\ +\frac{\phi(b_i)-\phi(-b_i)+b_j(\phi_\downarrow(b_i)-\phi_\downarrow(-b_i))}{2b_i}c_ja_{ik}\\ +\frac{\phi_\downarrow(b_i)-\phi_\downarrow(-b_i)+b_j(\phi_{\downarrow\downarrow}(b_i)-\phi_{\downarrow\downarrow}(-b_i))}{2b_i}\delta a_{ij}a_{ik}$$

$a_{kj}\mapsto$

$a_{ij}\mapsto a_{ij}$

To do. • Perhaps I should find a way to highlight the fact that v is a perturbation of w. • Position FiC. • Position the 2D Lie bialgebras.

Recycling.