Cheat Sheet β

verification at 2014-06/CheatSheetBeta-Verification.nb, 2014-05/GoodFormulas/Demo.nb
http://drorbn.net/AcademicPensieve/2015-04/; initiated 24/3/13; continues 2014-07; modified 18/5/15, 10:52am; continued Projects/MetaCalculi

σ calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $\quad tm^{uv}_w = (T_u, T_v \to T_w)$, $\quad hm^{xy}_z : \sigma \mapsto (\sigma\backslash\{x,y\}) \cup (z \to \sigma_x\sigma_y)$, $\quad tha^{ux} = I$, $\quad R^\pm_{ux} \mapsto T_u^{\pm 1}$

β-calculus. Constraints. • Sum of column x is $\sigma_x - 1$. • At $T_* = 1$, $\omega = 1$ and $A = 0$.

$$\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & A_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & A_2 \end{array} \underset{\beta}{=} \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ T_2 & 0 & A_2 \end{array} \qquad \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \Xi \end{array} \xrightarrow[T_u,T_v\to T_w]{\beta::tm^{uv}_w} \begin{array}{c|c} \omega & H \\ \hline w & \alpha+\beta \\ T & \Xi \end{array} \qquad \begin{array}{c|ccc} \omega & x & y & H \\ \hline T & \alpha & \beta & \Xi \end{array} \xrightarrow[\beta]{hm^{xy}_z} \begin{array}{c|cc} \omega & z & H \\ \hline T & \alpha+\sigma_x\beta & \Xi \end{array}$$

$$\begin{array}{c|cc} \omega & x & H \\ \hline u & \alpha & \theta \\ T & \phi & \Xi \end{array} \xrightarrow[\substack{\beta \\ \nu:=1+\alpha}]{tha^{ux}} \begin{array}{c|cc} \nu\omega & x & H \\ \hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu \\ T & \phi/\nu & \Xi - \phi\theta/\nu \end{array} \qquad \rho^\pm_{ux} \underset{\beta}{=} \begin{array}{c|c} 1 & x \\ \hline u & T_u^{\pm 1} - 1 \end{array}$$

Gassner calculus Γ. Preserves $C_1 :=$ [col sum = 1] ($\Leftrightarrow$OC) and ✓ $C_2 := [\forall a, b,\ (T_a - 1) \mid (A_{ab} - \delta_{ab}\sigma_b)]$
• Except under tr_c, at $T_* = 1$, $\omega = 1$ and $A = I$.

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\substack{\mu:=1-\beta \\ T_a,T_b\to T_c}]{m^{ab}_c} \begin{array}{c|cc} \mu\omega & c & S \\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu \\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu \end{array} \qquad \begin{array}{c|cc} \omega & c & S \\ \hline c & \alpha & \theta \\ S & \psi & \Xi \end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c} \mu\omega & S \\ \hline S & \Xi+\psi\theta/\mu \end{array} \qquad R^\pm_{ab} \underset{\gamma}{=} \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array}$$

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu:=T_a-1 \\ \nu:=\alpha-\sigma_a}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_a T_a - \nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array} \right)_{T_a\mapsto T_bT_c}$$

Satisfies: ✓ $R^+_{13} /\!\!/ q\Delta^1_{12} = R^+_{23} \# R^+_{13}$.
✓ $R^-_{13} /\!\!/ q\Delta^1_{12} = R^-_{13} \# R^-_{23}$.
✓ $q\Delta^a_{a_1a_2} /\!\!/ q\Delta^b_{b_1b_2} /\!\!/ m^{a_1b_1}_{c_1} /\!\!/ m^{a_2b_2}_{c_2} = m^{ab}_c /\!\!/ q\Delta^c_{c_1c_2}$.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow{dS^a} \left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array} \right)_{T_a\to T_a^{-1}}$$

Satisfies: ✓ $R^\pm_{12} /\!\!/ dS^{1 \text{ or } 2} = R^\mp_{12}$. ✓ $dm^{ab}_c /\!\!/ dS^c = dS^a /\!\!/ dS^b /\!\!/ dm^{ba}_c$.
✓ $dS^a /\!\!/ dS^a = I$. ✓ $q\Delta^a_{bc} /\!\!/ dS^b /\!\!/ dS^c = dS^a /\!\!/ q\Delta^a_{cb}$.
✓ Assuming C_2, $d\eta^a /\!\!/ d\epsilon_a = q\Delta^a_{bc} /\!\!/ dS^c /\!\!/ dm^{bc}_a$ (also 3 variants).

The map (tangle $T \mapsto$ matrix A) is anti-multiplicative. The MVA mod units: $L \mapsto (\omega, A) \mapsto \omega\det'(A-I)/(1-T')$ ✓

Burau. On $b \in uB_n$, Bu: $\sigma_i^{\pm 1} \mapsto U_i^{\pm 1}$. **Unitarity.** With $U = Bu(b)$, $\bar{U}\Omega_n U^T = \Omega_n$.

$$\textbf{Thm. } \Gamma(b) = \begin{array}{c|ccc} 1 & s_{b(1)} & s_{b(2)} & \cdots \\ \hline s_1 & & & \\ s_2 & & Bu(b)^T & \\ \vdots & & & \end{array}. \qquad U_i = \begin{pmatrix} I_i & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}, U_i^{-1} = \begin{pmatrix} I_i & & & \\ & 0 & 1 & \\ & \bar{t} & 1-\bar{t} & \\ & & & I_{n-i-1} \end{pmatrix}, \Omega_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1-t & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & \cdots & 1 \end{pmatrix}$$

Some matrices: $\begin{pmatrix} 1-t_i & 1 \\ t_i & 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 \\ 1 & 1 & \frac{1}{1-t_3} \end{pmatrix}$, $\begin{pmatrix} 1-t_j & 1 \\ t_i & 0 \end{pmatrix}$, $\begin{pmatrix} -\frac{t_1-1}{t_1} & 0 & 0 \\ \frac{(t_1-1)(t_2-1)}{t_2} & -\frac{t_2-1}{t_2} & 0 \\ \frac{(t_1-1)(t_3-1)}{t_3} & \frac{(t_2-1)(t_3-1)}{t_3} & -\frac{t_3-1}{t_3} \end{pmatrix}$

To do. • Full verification program. • R1? • Precise relation with Burau/Gassner. • Concordance. • Unitarity. • Planarity. • A depth-mirror property for u-objects. • Mutations? • Link relations? • Behaviour of A/MVA under mirror/strand reversal?

β-better calculus. Constraints. • Sum of column x is $(\sigma_x - 1)\omega$. • $\omega^{k-1} \mid \Lambda^k A$. • At $T_* = 1$, $\omega = 1$ and $A = 0$.

$$\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array} \underset{\beta_b}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & \omega_2 A_1 & 0\\ T_2 & 0 & \omega_1 A_2\end{array}$$

$$\begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \gamma\end{array} \xrightarrow[\beta_b]{tm_w^{uv}} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \gamma\end{array}\right)_{T_u,T_v\to T_w} \qquad \rho^\pm_{ux} \underset{\beta_b}{=} \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}$$

$$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \gamma\end{array} \xrightarrow[\beta_b]{hm_z^{xy}} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \gamma\end{array}$$

$$\begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \beta\\ T & \gamma & \delta\end{array} \xrightarrow[\beta_b]{tha^{ux}} \begin{array}{c|cc}\omega+\alpha & x & H\\ \hline u & \sigma_x\alpha & \sigma_x\beta\\ T & \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{array} =: \begin{array}{c|c}\cdot & -\\ \hline | & \begin{pmatrix}\sigma_x & 0\\ 0 & 1\end{pmatrix}\cdot A^{ux}\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\beta_b \checkmark]{m_c^{ab}} \left(\begin{array}{c|cc}\omega+\beta & c & S\\ \hline c & \gamma+\sigma_a\delta+\sigma_b(\alpha+\sigma_a\beta)+\frac{\beta\gamma-\alpha\delta}{\omega} & \epsilon+\sigma_b\theta+\frac{\beta\epsilon-\delta\theta}{\omega}\\ S & \phi+\sigma_a\psi+\frac{\beta\phi-\alpha\psi}{\omega} & \Xi+\frac{\beta\Xi-\psi\theta}{\omega}\end{array}\right)_{T_a,T_b\to T_c}$$

The MVA (mod units): n-component $L \mapsto (\sigma,\omega,A) \mapsto \omega^{2-n}\det'(A-\omega\,\mathrm{diag}((\sigma_i-1))/(1-T')$ ✓

Note. $A^{ux} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \frac{(\omega+\alpha)\delta-\gamma\beta}{\omega}\end{pmatrix} = \frac{1}{\omega}\left[(\omega+\alpha)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} - \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix}\right] = \frac{1}{\omega}\left[(\omega+a_{ux})A - a_{*x}a_{u*}\right].$

Claim. $\omega^{k-1} \mid \Lambda^k A$ and $\omega^k \mid \Lambda^{k+1}A$ implies $(\omega+\alpha)^{k-1} \mid \Lambda^k A^{ux}$, with $\alpha = a_{ux}$.

Proof. With $\bar u \in T^k$ and $\bar x \in H^k$, ω^k divides $\begin{vmatrix}\omega & 0\\ 0 & a_{\bar u\bar x}\end{vmatrix}$ and $\begin{vmatrix}a_{ux} & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix}$ and hence their sum, $\begin{vmatrix}\omega+\alpha & a_{u\bar x}\\ a_{\bar u x} & a_{\bar u\bar x}\end{vmatrix} = (\omega+\alpha)\begin{vmatrix}1 & 0\\ 0 & a_{\bar u\bar x}-\frac{1}{\omega+\alpha}a_{\bar ux}a_{u\bar x}\end{vmatrix} = \frac{1}{(\omega+\alpha)^{k-1}}\left|(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right|$. So $\frac{1}{(\omega+\alpha)^{k-1}}\left|\frac{1}{\omega}\left[(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right]\right|$ is integral. □

That is, with $A_{\bar u;\bar x}$ denoting minors, if $\omega^{k-1}\mu_{\bar u;\bar x} = A_{\bar u;\bar x}$ and $\omega^k\mu_{u\bar u;x\bar x} = A_{u\bar u;x\bar x}$, then $(\omega+\alpha)^{k-1}(\mu_{\bar u;\bar x}+\mu_{u\bar u;x\bar x}) = A^{ux}_{\bar u;\bar x}$.

Λ-calculus. $\Lambda(T;H) = R(T)\otimes(\Lambda(T)\otimes\Lambda(H))_=$, with $R(T)$ Laurent polynomials in $\{T_u\}_{u\in T}$. $\lambda_1 * \lambda_2 = \lambda_1(\wedge\otimes\wedge)\lambda_2$

$tm_w^{uv}:\ u,v\to w,\ T_u,T_v\to T_w$ $\quad hm_z^{xy}:\ x\to z,\ y\to\sigma_x z$ $\quad tha^{ux}:\ \lambda\mapsto(1+i_u\otimes i_x)\lambda /\!/ (u\to\sigma_x u)$ $\quad \rho^\pm_{ux} = 1+(T_u^{\pm1}-1)ux$

Relations. • $\rho^+_{ux}\rho^-_{vy} /\!/ tm_w^{uv} /\!/ hm_z^{xy} = t\epsilon_w h\epsilon_z$. • $\rho^{s_1}_{ux}\rho^{s_2}_{vy}\rho^{s_2}_{wz} /\!/ tm_v^{vw} /\!/ hm_x^{xy} /\!/ tha^{ux} = \rho^{s_2}_{vx}\rho^{s_2}_{wz}\rho^{s_1}_{uy} /\!/ tm_v^{vw} /\!/ hm_x^{xy}$.