Kapovitch's class, Fri March 20: Rational Homotopy Theory

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Q. Given X, Y, when are they h.e. (homotopy equiv)?

<u>Whitehead's</u> <u>thm</u> If X, Y are CW complexes and $\phi: X \to Y$ induces isomorphisms on homotopy groups, then X & Y are h.e.

$\phi: X \to Y$ induces isomorphisms of π_i's is "weak homotopy equivalence".

<u>Def</u> $f: X \to Y$ is a "rational h.e." if

$f_*: \pi_1 X \xrightarrow{\sim} \pi_1 Y$ &

$f_{*i}: \pi_i(X) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_i(Y) \otimes \mathbb{Q}$.

Examples 1. h.e. is rational h.e.

2. $f: S^n \to S^n$ (for $n>1$) has non-zero degree is a rational h.e.. E.g., $A \to A^2$ as a map $SU(2) = S^3 \to SU(2) = S^3$.

(Follows from Hurwicz thm: If X is $(n-1)$-connected, then $Hu: \pi_n \to H_n$ is an iso.)

Also, if $f: X \to Y$, X, Y simply-connected, $f_*: H_i(X) \to H_i(Y)$ is iso $\forall i$, then $f_*: \pi_i(X) \to \pi_i(Y)$ is iso $\forall i$.

Everything we do from here on is assuming simple-connectivity.
[Though rational homotopy generalizes to spaces w/ nilpotent π_1]

A $\mathbb{Q}$-version: If $f: X \to Y$ between s.c. spaces is rationally iso on homology, then same is true on homotopy.

(Yet Whitehead's on $\mathbb{Q}$ is false)

<u>Def'n</u> We say that X is rationally equiv. to Y if there is a chain of maps

$$X = X_0 \to X_1 \leftarrow X_2 \to \dots \to X_n = Y$$

s.t. all steps are rational equivalnces.

$X \longrightarrow C^*(X)$ DGA of simplicial cochains

<u>Def</u> A map $\phi: A \to B$ of DGA's is

a quasi-isomorphism if

$f_*: H^*(A) \xrightarrow{\sim} H^*(B)$. [qi's do not nec. have inverse qi's]

Thm $X \simeq Y$ iff $C^*(X,\mathbb{Q}) \simeq C^*(Y,\mathbb{Q})$

rationally equiv.

in the same zig-zag sense as before

Aside $H^*_{sing}(M,\mathbb{R}) \cong H^*_{dR}(M,\mathbb{R})$ for manifolds M.

Thm Given s.c. X, there exists a commutative DGA $A_{PL}(X)$ qi to $C^*(X,\mathbb{Q})$ & $X \to A_{PL}(X)$ is a functor.

Idea $X \longrightarrow \tilde{X} \longrightarrow \Omega_{PL}(\tilde{X}) \longrightarrow A_{PL}$ *

simplicial approx. of X: take an abstract simplex for every physical simplex in X, & glue when appropriate.

formal poly. diff. forms on an abstract simp. Complex.

*: The last step is an algebraic "approximation by commutative" Thm (?).

Main Thm $X \sim Y$ iff $A_{PL}(X) \sim A_{PL}(Y)$

rationally

by a chain of qi's.