Odesskii @ Winter 2015 Master Class: Elliptic Algebras and Poisson Structures

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over $\mathbb{C}$

PBW Algebra: V: v.s. of dim n.

$L \subset V \otimes V$

$A = TV/\langle L \rangle$ a quadratic algebra

graded

$$A = \underset{A_0}{\mathbb{C}} \oplus \underset{A_1}{V} \oplus \underset{A_2}{\frac{V\otimes V}{L}} \oplus \underset{A_3}{\frac{V\otimes V \otimes V}{V\otimes L + L \otimes V}} \oplus \cdots$$

Def A is a PBW algebra if

$$\dim A_m = \frac{n(n+1)\cdots(n+m-1)}{m!}.$$

Q: When is grG a PBW algebra?

Probably does not imply that $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ is a basis, but the latter is true in all practical examples.

Example 1. Skew polynomials:

Generators $x_1 \dots x_n$

Relations: $x_j x_i = q_{i,j} x_i x_j \quad i<j \quad q_{ij} \neq 0$

Example 2. Generators $x_1 \dots x_n, z$

Relations: $x_i x_j - x_j x_i = C_{ij}^k x_k z$

$x_k z = z x_k$

structure constants of a Lie algebra.

Example 3. Drinfeld Algebra:

(is it gr of something?)

Generators $x_1 \dots x_n$

Relations: For $1 \le k, l < n$

$$x_{k+1}x_l - q x_l x_{k+1} = q x_k x_{l+1} - x_{l+1} x_k$$

To understand: first take $l=k$, then $l=k+1$, ...

$Q_{n,k}(\tau, \eta)$ $\quad (\tau,\eta) \sim \left(-\frac{1}{\tau}, \frac{\eta}{\tau}\right) \sim (\tau+1, \eta) \sim$

$\mathrm{Im}(\tau) > 0$ $\quad \sim (\tau, \eta+1) \sim (\tau, \eta+\tau)$

$\eta \in \mathbb{C}$

$n \geq 3 \quad 1 \leq k < n, \quad (n,k) = 1.$

Suppose we have a family L_h, with $L_0 = \Lambda^2 V$.

Let $h \neq 0$ but $h^2 = 0$:

$$L_h = \{x_i x_j - x_j x_i + h C_{ij}^{kl} x_k x_l\}$$

A PBW ring over $\mathbb{C}[h]/h^2=0$.

Get:
$$h x_i x_j = h x_j x_i$$

$\{x_i, x_j\} := C_{ij}^{kl} x_k x_l$ is a Poisson structure.

<u>Examples</u> skew polys $x_i x_j = q_{ij} x_j x_i$, $q_{ij} = 1 + h_{ij}$

$h_{ij} = -h_{ji}$

$\Rightarrow \{x_i, x_j\} = h_{ij} x_i x_j$ is a Poisson structure.

<u>Drinfel'd Algebra</u>: Take $q = 1+h$, get

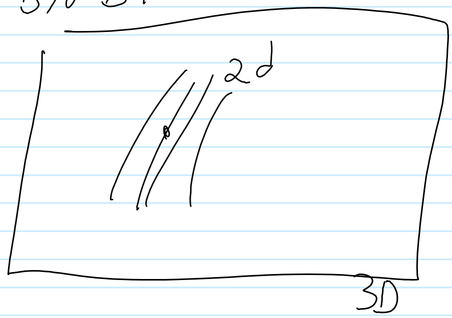
$$\{x_i, x_j\} = \text{exercise.}$$

Hint: start with $\{x_i, x_{i+1}\}$, $\{x_i, x_{i+2}\}$, ...

Poisson structures in low dimensions:

x, y, z

$p(x, y, z) = c$



$$0 = \{x, p\} = \frac{\partial p}{\partial y}\{x, y\} - \frac{\partial p}{\partial z}\{z, x\}$$

so $\{x, y\} = \Lambda \frac{\partial p}{\partial z}$ for some $\Lambda(x, y)$.

$\{y, z\} = \Lambda \frac{\partial p}{\partial x}$

$\{z, x\} = \Lambda \frac{\partial p}{\partial y}$

This gives a Poisson structure for any (p, Λ)!

<u>Example</u> $p = x^a y^b z^c$ $\quad a, b, c \in \mathbb{C}$

$$\Lambda = (x^{a-1} y^{b-1} z^{c-1})^{-1}$$

$\{x, y\} = cxy \quad \{y, z\} = ayz \quad \{z, x\} = bxz$

<u>Example</u> $\Lambda = 1$, p cubic poly to get a quadratic P.S.:

$p = \frac{1}{3}(x^3 + y^3 + z^3 + 3kxyz)$ (most general possible after a linear change)

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$\Longrightarrow$ $\{x, y\} = kxy + z^2$
$\{y, z\} = kyz + x^2$
$\{z, x\} = kxz + y^2$

Symplectic leaves are $p = c \in \mathbb{C}$

especially interesting: $\{p=0\} \setminus \{x=y=z=0\}$

Q: what do we know about the variety

$x^3 + y^3 + z^3 + kxyz = 0$

it is the cone over the image of

$z \mapsto (\theta_0(z), \theta_1(z), \theta_2(z))$ a basis on

$$\Theta_3(\tau) = \left\{ f : \begin{array}{l} f(z+1) = f(z) \\ f(z+\tau) = -e^{-2\pi i 3z} f(z) \end{array} \right\}$$

$k = k(\tau)$ some modular function.

$f \in \Theta_3 \mapsto f(z + \frac{1}{3}) \in \Theta_3$