# Filmus: Analysis of Boolean functions on non-product domains

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1. What is analysis of Bool functns?
2. Concrete example: Lovász proof of Erdős–Ko–Rado. (EKR)
3. Non-product domains: symmetric group & slice.

A Boolean function is $f: \mathcal{D} \to \{0,1\}$

finite w/ prob measure, often $\{0,1\}^n$ "Boolean cube" w/ uniform measure or

$\mu_p(x) = p^{x_1} \cdots p^{x_n} \cdot (1-p)^{\Sigma(1-x_i)}$

Basic ideas:

1. Extending the range: $f: \mathcal{D} \to \{0,1\}$ is also $f: \mathcal{D} \to \mathbb{R}, \mathbb{C}$
2. Extending the domain: $f: \mathcal{D} \to \{0,1\}$ becomes

$$\tilde{f}: \tilde{\mathcal{D}} \to \mathbb{R}$$

w/ $\tilde{\mathcal{D}} \supset \mathcal{D}$ & $f$ & $\tilde{f}$ having similar props.

Sample applications:

1. Extremal combinatorics, EKR Thm
2. Random graphs, Friedgut's sharp threshold thm.

3. Voting theory — majority is stablest.

4. TCS: Hardness of approx., property testing.

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Thm (EKR): Let $F \subset \binom{[n]}{k}$ "slice" be intersecting ($A, B \in F \Rightarrow A \cap B \neq \phi$). If $k < \frac{n}{2}$, a star is "all sets containing a specific element"; $|\text{star}| = \binom{n-1}{k-1}$.

① Then $|F| \leq \binom{n-1}{k-1}$, & ② bound is achieved only by stars.

③ stability: If $|F|$ is near maximal, F is near a star. (Frankel, 80s)

Lovász proof

There exists an $\binom{[n]}{k} \times \binom{[n]}{k}$ matrix $A$ s.t. ① $A_{ST} = 1$ iff $S \cap T \neq \phi$.

② $\|A\| = \binom{n-1}{k-1}$, eigenspace of max eigval $\binom{n-1}{k-1}$ is spanned by stars.

Now if $F \subset \binom{[n]}{k}$ is intersecting & $f = 1_F$, then

$f^T A f = |F|^2$

b. $f^T A f \leq \|A\| \|f\|^2 = \binom{n-1}{k-1} |F|$

proving ① & ②

What's A?

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Extending the domain:

$$\binom{[n]}{k} \approx \mu_{k/n}$$

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Some EKR-like results that can only be proven this way?

1. EKR for v's.
2. EKR for permutations
3. EKR for $\Delta$-intersecting families of graphs.

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A. Friedgut-Kalai-Naor

Let $f: \{0,1\}^n \to \{0,1\}$ (unif. measure)

Easy: Suppose

$$f = c_0 + \sum_i c_i x_i$$

Then $f$ depends on only one variable "dictator".

Harder: Suppose $f$ is $L^2$-close to $c_0 + \sum c_i x_i$. Then $f \approx$ dictatorship

B. Kindler–Safra: Let $f:\{0,1\}^n \to \{0,1\}$

Easy: If $f$ is a degree $d$ poly, then $f$ is a $K_d$-junta (depends on $k_d$-coords, for some $k_d$)

Hard: stability.

C. Friedgut Junta Thm: $f:\{0,1\}^n \to \{0,1\}$

Let $\partial f = \{(x,y)\in\{0,1\}^n : |x-y|=1,\ f(x)\neq f(y)\}$

Easy: If $f$ is a $C$-junta, then

$$\frac{|\partial f|}{2^n} \leq C$$

Hard: Converse

D. Invariance principle: Let $f$ be a "nice" poly then $f$ evaluated on uniform is near $f$ evaluated on Gaussians.

| | $S_n$ | $\binom{[n]}{k}$ |
|---|---|---|
| FKN | Ellis, Friedgut, Filmus | Filmus |
| KS | EFF | ? |
| Junta | probably False | Wimmer, Filmus |
| Inv | ? | F, Kindler, Mossel, Wimmer. |

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