

Pensieve header: Infinitesimal Elliptic computations, version Associative.

```

AllWords[n_, d_] := AW@@@Tuples[Union@@Table[{x_i, y_i}, {i, n}], d]

AllRelators2[n_] := Flatten[{
  Table[{cx_{i,j}, cy_{i,j}, cxy_{i,j}}, {i, 1, n-1}, {j, i+1, n}],
  Table[c_{n,i}, {i, n}]
}];

AllRelators3[n_] := Flatten[Table[
  {ctx_{i,j,k}, cty_{i,j,k}, ctx_{j,k,i}, cty_{j,k,i}, ctx_{k,i,j}, cty_{k,i,j}},
  {i, 1, n-2}, {j, i+1, n-1}, {k, j+1, n}
]];

AllRelators2[n_, d_] := Flatten[Table[
  Outer[Join, AllWords[n, k], AW /@ AllRelators2[n], AllWords[n, d-k-2]],
  {k, 0, d-2}
]];

AllRelators3[n_, d_] := Flatten[Table[
  Outer[Join, AllWords[n, k], AW /@ AllRelators3[n], AllWords[n, d-k-3]],
  {k, 0, d-3}
]];

AllRelators[n_, d_] := Join[AllRelators2[n, d], AllRelators3[n, d]];

b[x_, y_] := AW[x, y] - AW[y, x];
b[x_, y_, z_] := AW[x, y, z] - AW[y, x, z] - AW[z, x, y] + AW[z, y, x];
diff[x_i_] := x_i; diff[y_i_] := y_i;
diff[cx_{i,j}] := b[x_i, x_j];
diff[cy_{i,j}] := b[y_i, y_j];
diff[cxy_{i,j}] := b[x_i, y_j] - b[x_j, y_i];
diff[ctx_{i,j,k}] := b[x_i, y_j, x_k];
diff[cty_{i,j,k}] := b[x_i, y_j, y_k];
diff[c_{n,i}] :=  $\sum_{j=1}^n b[x_i, y_j]$ ;

diff[aw_AW] :=
  Distribute[diff /@ aw] /. AW[lft___, c_. * AW[w___], rgt___] => c * AW[lft, w, rgt]

```

```

Dim[n_, d_] := Module[{im, v, vc},
  im =
  (diff /@ AllRelators[n, d]) /. Thread[AllWords[n, d] → Table[v[i], {i, (2 n)^d}]];
  If[im == {},
    (2 n)^d,
    (2 n)^d - MatrixRank[
      SparseArray[
        Flatten[Table[
          (*Cases[{im[[i]]}, c_.*v[j_] → ({i,j}→c), ∞], {i, Length[im]}]
          Wrong! c_.*v[j_] matches some terms twice,
          producing extra matrix entries. *)
          vc = im[[i]];
          ({i, #[[1]]} → Coefficient[vc, #]) & /@ Cases[{vc}, _v, ∞],
          {i, Length@im}
        ]],
        {Length[im], (2 n)^d}
      ]
    ]
  ]
]

Table[Dim[2, d], {d, 6}] // Timing
{72.134862, {4, 11, 26, 57, 120, 247}}

```

## Testing

```
AllWords[4, 0]
```

```
{AW[]}
```

```
AllWords[2, 3] // Length
```

```
64
```

```
AllRelators2[3]
```

```
{cx1,2, cy1,2, cxy1,2, cx1,3, cy1,3, cxy1,3, cx2,3, cy2,3, cxy2,3, c3,1, c3,2, c3,3}
```

```
AllRelators3[4]
```

```
{ctx1,2,3, cty1,2,3, ctx2,3,1, cty2,3,1, ctx3,1,2, cty3,1,2, ctx1,2,4, cty1,2,4,
ctx2,4,1, cty2,4,1, ctx4,1,2, cty4,1,2, ctx1,3,4, cty1,3,4, ctx3,4,1, cty3,4,1,
ctx4,1,3, cty4,1,3, ctx2,3,4, cty2,3,4, ctx3,4,2, cty3,4,2, ctx4,2,3, cty4,2,3}
```

**{AllRelators2[3, 3], AllRelators3[3, 3]}**

{AW[cx<sub>1,2</sub>, x<sub>1</sub>], AW[cx<sub>1,2</sub>, x<sub>2</sub>], AW[cx<sub>1,2</sub>, x<sub>3</sub>], AW[cx<sub>1,2</sub>, y<sub>1</sub>], AW[cx<sub>1,2</sub>, y<sub>2</sub>], AW[cx<sub>1,2</sub>, y<sub>3</sub>],  
 AW[cy<sub>1,2</sub>, x<sub>1</sub>], AW[cy<sub>1,2</sub>, x<sub>2</sub>], AW[cy<sub>1,2</sub>, x<sub>3</sub>], AW[cy<sub>1,2</sub>, y<sub>1</sub>], AW[cy<sub>1,2</sub>, y<sub>2</sub>], AW[cy<sub>1,2</sub>, y<sub>3</sub>],  
 AW[cxy<sub>1,2</sub>, x<sub>1</sub>], AW[cxy<sub>1,2</sub>, x<sub>2</sub>], AW[cxy<sub>1,2</sub>, x<sub>3</sub>], AW[cxy<sub>1,2</sub>, y<sub>1</sub>], AW[cxy<sub>1,2</sub>, y<sub>2</sub>],  
 AW[cxy<sub>1,2</sub>, y<sub>3</sub>], AW[cx<sub>1,3</sub>, x<sub>1</sub>], AW[cx<sub>1,3</sub>, x<sub>2</sub>], AW[cx<sub>1,3</sub>, x<sub>3</sub>], AW[cx<sub>1,3</sub>, y<sub>1</sub>], AW[cx<sub>1,3</sub>, y<sub>2</sub>],  
 AW[cx<sub>1,3</sub>, y<sub>3</sub>], AW[cy<sub>1,3</sub>, x<sub>1</sub>], AW[cy<sub>1,3</sub>, x<sub>2</sub>], AW[cy<sub>1,3</sub>, x<sub>3</sub>], AW[cy<sub>1,3</sub>, y<sub>1</sub>], AW[cy<sub>1,3</sub>, y<sub>2</sub>],  
 AW[cy<sub>1,3</sub>, y<sub>3</sub>], AW[cxy<sub>1,3</sub>, x<sub>1</sub>], AW[cxy<sub>1,3</sub>, x<sub>2</sub>], AW[cxy<sub>1,3</sub>, x<sub>3</sub>], AW[cxy<sub>1,3</sub>, y<sub>1</sub>],  
 AW[cxy<sub>1,3</sub>, y<sub>2</sub>], AW[cxy<sub>1,3</sub>, y<sub>3</sub>], AW[cx<sub>2,3</sub>, x<sub>1</sub>], AW[cx<sub>2,3</sub>, x<sub>2</sub>], AW[cx<sub>2,3</sub>, x<sub>3</sub>],  
 AW[cx<sub>2,3</sub>, y<sub>1</sub>], AW[cx<sub>2,3</sub>, y<sub>2</sub>], AW[cx<sub>2,3</sub>, y<sub>3</sub>], AW[cy<sub>2,3</sub>, x<sub>1</sub>], AW[cy<sub>2,3</sub>, x<sub>2</sub>], AW[cy<sub>2,3</sub>, x<sub>3</sub>],  
 AW[cy<sub>2,3</sub>, y<sub>1</sub>], AW[cy<sub>2,3</sub>, y<sub>2</sub>], AW[cy<sub>2,3</sub>, y<sub>3</sub>], AW[cxy<sub>2,3</sub>, x<sub>1</sub>], AW[cxy<sub>2,3</sub>, x<sub>2</sub>],  
 AW[cxy<sub>2,3</sub>, x<sub>3</sub>], AW[cxy<sub>2,3</sub>, y<sub>1</sub>], AW[cxy<sub>2,3</sub>, y<sub>2</sub>], AW[cxy<sub>2,3</sub>, y<sub>3</sub>], AW[c<sub>1</sub>, x<sub>1</sub>], AW[c<sub>1</sub>, x<sub>2</sub>],  
 AW[c<sub>1</sub>, x<sub>3</sub>], AW[c<sub>1</sub>, y<sub>1</sub>], AW[c<sub>1</sub>, y<sub>2</sub>], AW[c<sub>1</sub>, y<sub>3</sub>], AW[c<sub>2</sub>, x<sub>1</sub>], AW[c<sub>2</sub>, x<sub>2</sub>], AW[c<sub>2</sub>, x<sub>3</sub>],  
 AW[c<sub>2</sub>, y<sub>1</sub>], AW[c<sub>2</sub>, y<sub>2</sub>], AW[c<sub>2</sub>, y<sub>3</sub>], AW[c<sub>3</sub>, x<sub>1</sub>], AW[c<sub>3</sub>, x<sub>2</sub>], AW[c<sub>3</sub>, x<sub>3</sub>], AW[c<sub>3</sub>, y<sub>1</sub>],  
 AW[c<sub>3</sub>, y<sub>2</sub>], AW[c<sub>3</sub>, y<sub>3</sub>], AW[x<sub>1</sub>, cx<sub>1,2</sub>], AW[x<sub>1</sub>, cy<sub>1,2</sub>], AW[x<sub>1</sub>, cxy<sub>1,2</sub>], AW[x<sub>1</sub>, cx<sub>1,3</sub>],  
 AW[x<sub>1</sub>, cy<sub>1,3</sub>], AW[x<sub>1</sub>, cxy<sub>1,3</sub>], AW[x<sub>1</sub>, cx<sub>2,3</sub>], AW[x<sub>1</sub>, cy<sub>2,3</sub>], AW[x<sub>1</sub>, cxy<sub>2,3</sub>], AW[x<sub>1</sub>, c<sub>1</sub>],  
 AW[x<sub>1</sub>, c<sub>2</sub>], AW[x<sub>1</sub>, c<sub>3</sub>], AW[x<sub>2</sub>, cx<sub>1,2</sub>], AW[x<sub>2</sub>, cy<sub>1,2</sub>], AW[x<sub>2</sub>, cxy<sub>1,2</sub>], AW[x<sub>2</sub>, cx<sub>1,3</sub>],  
 AW[x<sub>2</sub>, cy<sub>1,3</sub>], AW[x<sub>2</sub>, cxy<sub>1,3</sub>], AW[x<sub>2</sub>, cx<sub>2,3</sub>], AW[x<sub>2</sub>, cy<sub>2,3</sub>], AW[x<sub>2</sub>, cxy<sub>2,3</sub>], AW[x<sub>2</sub>, c<sub>1</sub>],  
 AW[x<sub>2</sub>, c<sub>2</sub>], AW[x<sub>2</sub>, c<sub>3</sub>], AW[x<sub>3</sub>, cx<sub>1,2</sub>], AW[x<sub>3</sub>, cy<sub>1,2</sub>], AW[x<sub>3</sub>, cxy<sub>1,2</sub>], AW[x<sub>3</sub>, cx<sub>1,3</sub>],  
 AW[x<sub>3</sub>, cy<sub>1,3</sub>], AW[x<sub>3</sub>, cxy<sub>1,3</sub>], AW[x<sub>3</sub>, cx<sub>2,3</sub>], AW[x<sub>3</sub>, cy<sub>2,3</sub>], AW[x<sub>3</sub>, cxy<sub>2,3</sub>],  
 AW[x<sub>3</sub>, c<sub>1</sub>], AW[x<sub>3</sub>, c<sub>2</sub>], AW[x<sub>3</sub>, c<sub>3</sub>], AW[y<sub>1</sub>, cx<sub>1,2</sub>], AW[y<sub>1</sub>, cy<sub>1,2</sub>], AW[y<sub>1</sub>, cxy<sub>1,2</sub>],  
 AW[y<sub>1</sub>, cx<sub>1,3</sub>], AW[y<sub>1</sub>, cy<sub>1,3</sub>], AW[y<sub>1</sub>, cxy<sub>1,3</sub>], AW[y<sub>1</sub>, cx<sub>2,3</sub>], AW[y<sub>1</sub>, cy<sub>2,3</sub>],  
 AW[y<sub>1</sub>, cxy<sub>2,3</sub>], AW[y<sub>1</sub>, c<sub>1</sub>], AW[y<sub>1</sub>, c<sub>2</sub>], AW[y<sub>1</sub>, c<sub>3</sub>], AW[y<sub>2</sub>, cx<sub>1,2</sub>], AW[y<sub>2</sub>, cy<sub>1,2</sub>],  
 AW[y<sub>2</sub>, cxy<sub>1,2</sub>], AW[y<sub>2</sub>, cx<sub>1,3</sub>], AW[y<sub>2</sub>, cy<sub>1,3</sub>], AW[y<sub>2</sub>, cxy<sub>1,3</sub>], AW[y<sub>2</sub>, cx<sub>2,3</sub>],  
 AW[y<sub>2</sub>, cy<sub>2,3</sub>], AW[y<sub>2</sub>, cxy<sub>2,3</sub>], AW[y<sub>2</sub>, c<sub>1</sub>], AW[y<sub>2</sub>, c<sub>2</sub>], AW[y<sub>2</sub>, c<sub>3</sub>], AW[y<sub>3</sub>, cx<sub>1,2</sub>],  
 AW[y<sub>3</sub>, cy<sub>1,2</sub>], AW[y<sub>3</sub>, cxy<sub>1,2</sub>], AW[y<sub>3</sub>, cx<sub>1,3</sub>], AW[y<sub>3</sub>, cy<sub>1,3</sub>], AW[y<sub>3</sub>, cxy<sub>1,3</sub>],  
 AW[y<sub>3</sub>, cx<sub>2,3</sub>], AW[y<sub>3</sub>, cy<sub>2,3</sub>], AW[y<sub>3</sub>, cxy<sub>2,3</sub>], AW[y<sub>3</sub>, c<sub>1</sub>], AW[y<sub>3</sub>, c<sub>2</sub>], AW[y<sub>3</sub>, c<sub>3</sub>]}  
 {AW[ctx<sub>1,2,3</sub>], AW[cty<sub>1,2,3</sub>], AW[ctx<sub>2,3,1</sub>], AW[cty<sub>2,3,1</sub>], AW[ctx<sub>3,1,2</sub>], AW[cty<sub>3,1,2</sub>]}

**{diff[cx<sub>2,4</sub>], diff[cxy<sub>2,4</sub>], diff[cty<sub>1,2,3</sub>], diff[c<sub>3,2</sub>]}**

{AW[x<sub>2</sub>, x<sub>4</sub>] - AW[x<sub>4</sub>, x<sub>2</sub>], AW[x<sub>2</sub>, y<sub>4</sub>] - AW[x<sub>4</sub>, y<sub>2</sub>] + AW[y<sub>2</sub>, x<sub>4</sub>] - AW[y<sub>4</sub>, x<sub>2</sub>],  
 AW[x<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>] - AW[y<sub>2</sub>, x<sub>1</sub>, y<sub>3</sub>] - AW[y<sub>3</sub>, x<sub>1</sub>, y<sub>2</sub>] + AW[y<sub>3</sub>, y<sub>2</sub>, x<sub>1</sub>],  
 AW[x<sub>2</sub>, y<sub>1</sub>] + AW[x<sub>2</sub>, y<sub>2</sub>] + AW[x<sub>2</sub>, y<sub>3</sub>] - AW[y<sub>1</sub>, x<sub>2</sub>] - AW[y<sub>2</sub>, x<sub>2</sub>] - AW[y<sub>3</sub>, x<sub>2</sub>]}

**diff[AW[cx<sub>1,2</sub>, x<sub>3</sub>]]**

AW[x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>] - AW[x<sub>2</sub>, x<sub>1</sub>, x<sub>3</sub>]

**diff /@ AllRelators[3, 3]**

**{n, d} = {3, 3}**

{3, 3}

**Dim[3, 3]**

75

**Dim[1, 1]**

2

**Dim[1, 2]**

3

**Dim[1, 3]**

4

**Dim[1, 5]**

6

**Dim[2, 1]**

4

**Dim[2, 2]**

11

**Dim[2, 3]**

26

**Dim[2, 4]**

57

**Dim[2, 5]**

120

**Dim[2, 6]**

247

**Dim[3, 1]**

6

**Dim[3, 2]**

24

**Dim[3, 3]**

75

**Dim[3, 4]**

192