Yampolsky's Class, Wed Jan 7: Conformal moduli of quadrilaterals and annuli, the uniformization theorem

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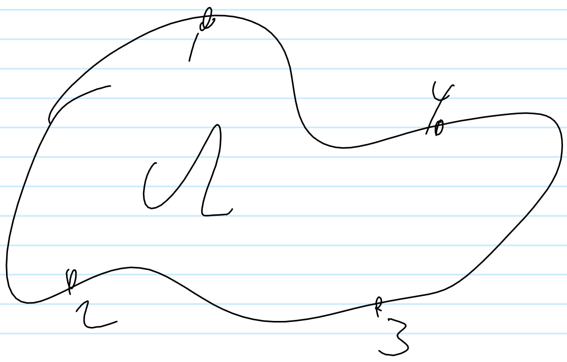
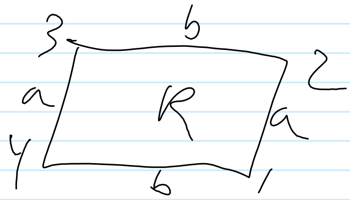
Conformal moduli of quadrilaterals
(the Grötzsch problem)

__Def__ A quadrilateral is a Jordan domain Ω with 4 marked points on the bdry

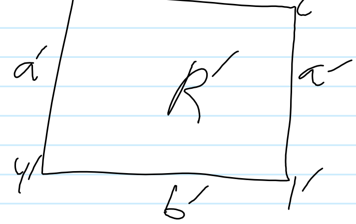


Two such Ω, Ω' are "conformally equiv." if $\exists$ conformal map $\Omega \to \Omega'$ mapping $\{1,2,3,4\} \to \{1',2',3',4'\}$ respecting the order.

__Example__ rectangles

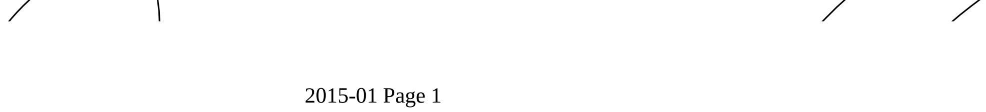


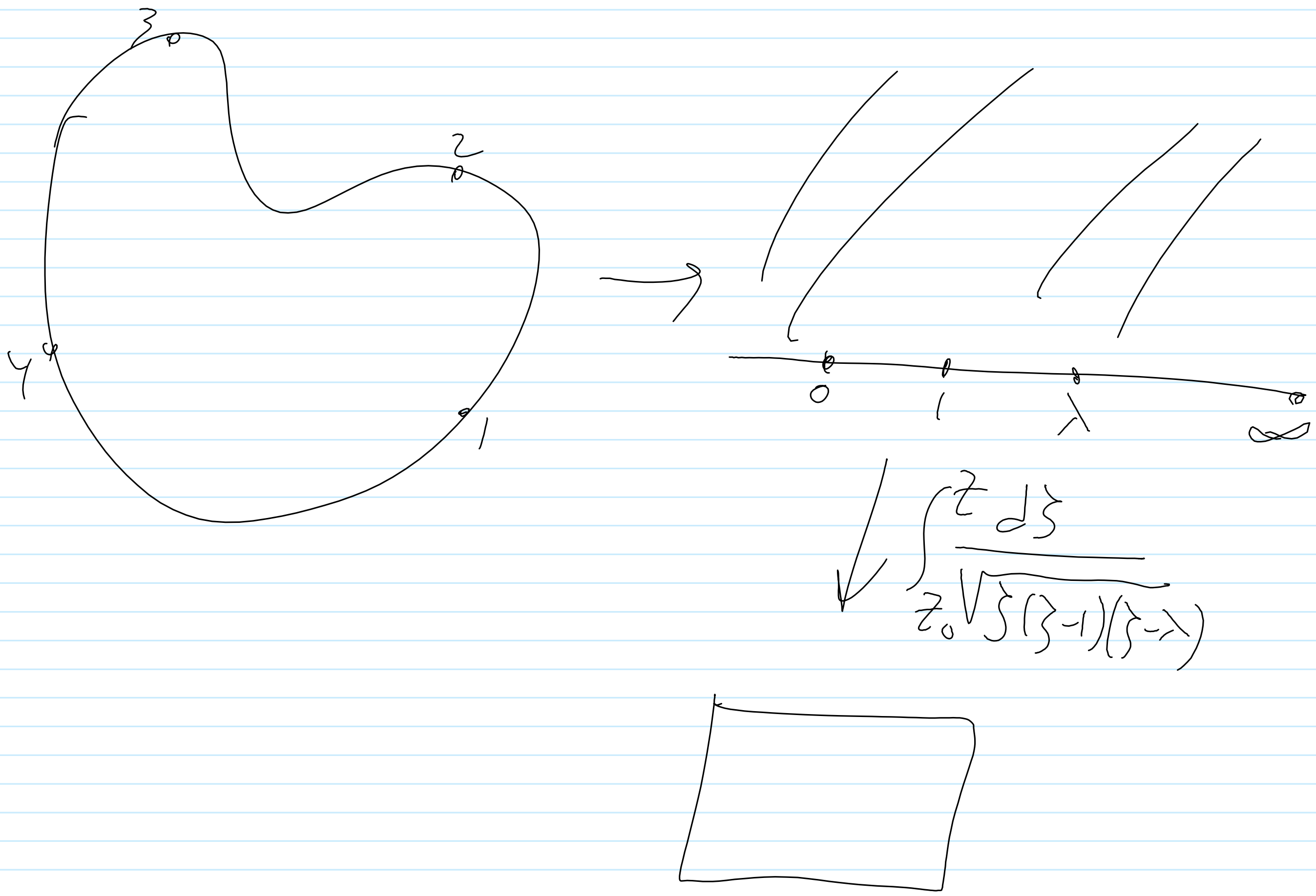
Two such are equiv. iff $\frac{b}{a} = \frac{b'}{a'}$ "the conformal modulus"



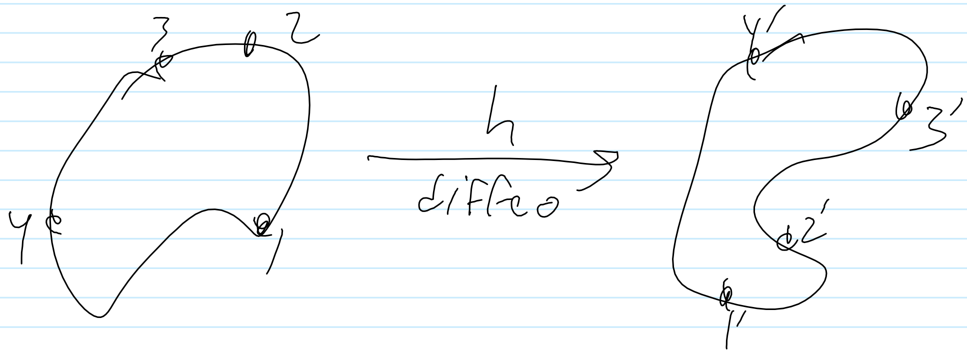
a positive real number.

steps to rectangular:

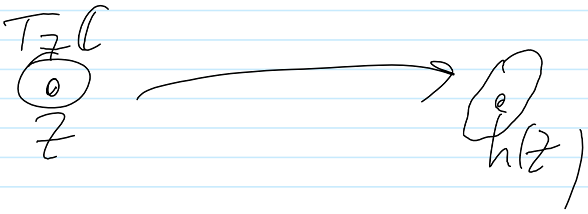




Next: define a metric between different conf. equiv. class



Idea: measure how far h is from being conformal.



def dilatation $K = \frac{\alpha}{\beta} \geq 1$

circles go to circles / ellipses

circles go to circles / ellipses

infinitesimal → circles; conformal, infinitesimal → circles; otherwise → ellipses

$K_h(z)$: dilatation of h at z.

$K_h := \sup_z K_h(z)$ (may be ∞)

Grötzsch: $\text{dist}(\Omega_1, \Omega_2) = \inf_h \log K_h$

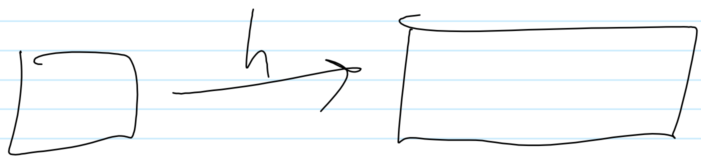
Triangle ineq. is easy. Not so trivial is that if dist$=0$ then Ω_1 & Ω_2 are conf. equiv.

Thm Grötzsch ineq:

$$\text{dist}(\Omega_1, \Omega_2) = \left| \log \frac{\text{mod}(\Omega_1)}{\text{mod}(\Omega_2)} \right|$$

← on moduli this is the Poincaré distance.

& $\exists$ unique diffeo which realizes the infimum, the real affine



C^1 diffeo $h : \Omega \to h(\Omega)$

$\Omega \subset \mathbb{R}^2_{x,y}$, $h(\Omega) \subset \mathbb{R}^2_{u,v}$

$(x,y) \longrightarrow (u,v)$

$z = x+iy \qquad w = u+iv$

$dz = dx + i\,dy$

$d\bar z = dy - i\,dy$

$$dh = dw = h_x dx + h_y dy = h_z dz + h_{\bar z} d\bar z$$

where $h_z = \frac{1}{2}(h_x - i h_y)$

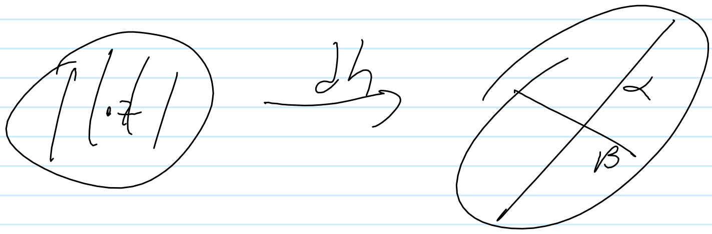
$$h_{\bar z} = \frac{1}{2}(h_x + i h_y)$$

$$h_z = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y)$$

$$h_{\bar z} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

($h_{\bar z} = 0$ if C-R eqn's)

The Jacobian of h: $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = |h_z|^2 - |h_{\bar z}|^2$



(area $= \alpha\beta$)

$$\mathrm{Jac}_z h = \alpha\beta = |h_z|^2 - |h_{\bar z}|^2$$

We want all maps to be orientation preserving $\Longleftrightarrow$

$$|h_z| \geq |h_{\bar z}|$$

aside

$$|h_z||dz| - |h_{\bar z}||d\bar z| \leq |dw| \leq |h_z||dz| + |h_{\bar z}||d\bar z|$$

max achieved at $\frac{h_z\, dz}{h_{\bar z}\, d\bar z} > 0 \Longleftrightarrow \arg \zeta = \frac{1}{2}\arg\frac{h_{\bar z}}{h_z}$

min at $\quad \cdots < 0 \Longleftrightarrow \arg\zeta = \frac{1}{2}\arg\frac{h_{\bar z}}{h_z} \pm \frac{\pi}{2}$

major axis is
minor —"— — . —

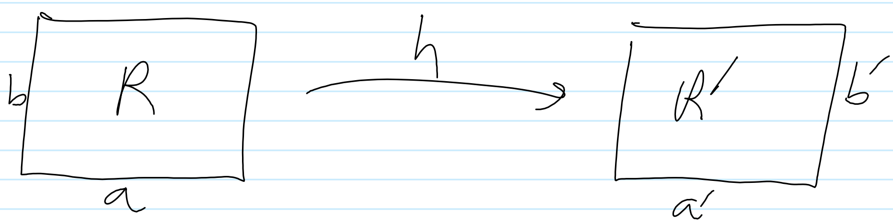
$$\Rightarrow K_h(z) = \frac{|h_z| + |h_{\bar z}|}{|h_z| - |h_{\bar z}|} \geq 1$$

Def the "complex dilatation"

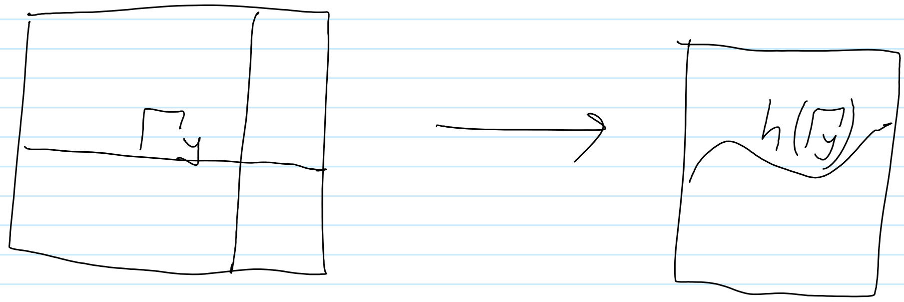
$$\mu(z) = \frac{h_{\bar z}}{h_z} \qquad |\mu| < 1$$

$$K_h(z) = \frac{1+|\mu|}{1-|\mu|}$$

Proof of Grötzsch: WLOG Ω & Ω' are rectangles



$$\inf_h \log K_h \stackrel{?}{=} \left|\log \frac{b/a}{b'/a'}\right|$$



Γ_y: curve horizontal at height y

$$\frac{a'}{a} \leq \frac{1}{|\Gamma_y|} \int_{\Gamma_y} |h_x| \, dx$$ "mean horizontal expansion"

so

$$\frac{1}{ab} \iint |h_x| \, dx \, dy \geq \frac{a'}{a}$$

so $$\left(\frac{a'}{a}\right)^2 \leq \left(\frac{1}{ab} \int_R |h_x| \, dx dy\right)^2$$

$$\leq \frac{1}{ab} \int |h_x|^2 \, dx dy = \#$$

using Cauchy-Schwartz

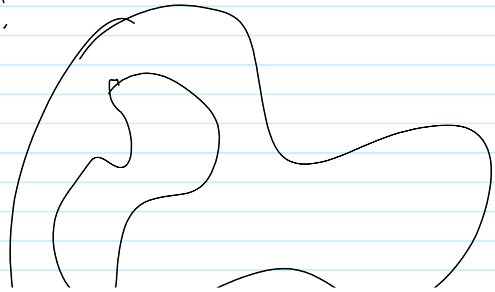
but $|h_x|^2 \leq \alpha^2 = \mathrm{Jac}_z(h) \cdot K_h(z)$

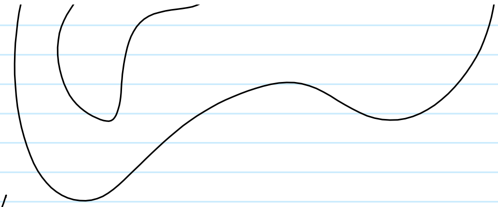
$$\leq \mathrm{Jac}_z h \cdot K_h$$

$$\# \leq \frac{1}{ab} K_h \iint \mathrm{Jac} = K_h \frac{a'b'}{ab}$$

so $K_h \geq$ desired thing.

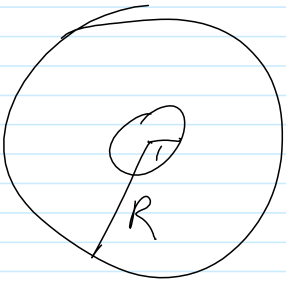
An equivalent conformal moduli problem is annuli:





$\underbrace{\text{Thm All}}_{}$ Koebe?

Thm All are conformally equiv. to the round case:



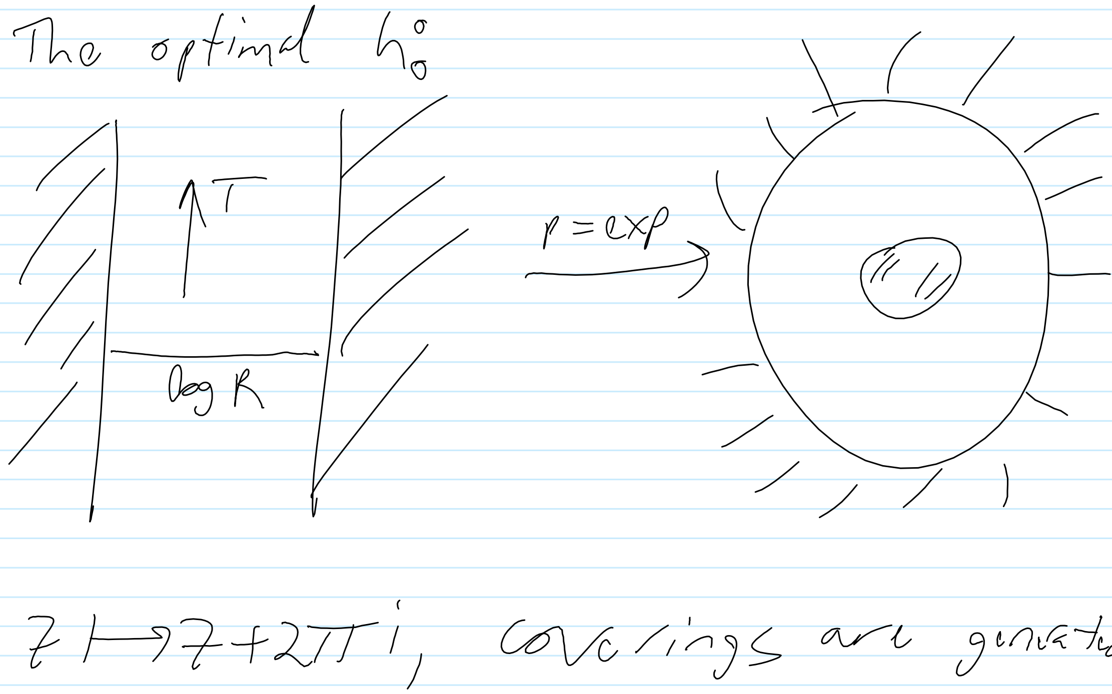
Claim $A_1 \overset{conf}{\sim} A_2 \iff R_1 = R_2$

$$\mathrm{mod}(A) := \frac{1}{2\pi} \log R > 0$$

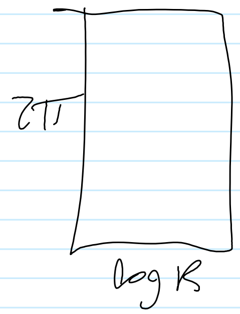
$$\mathrm{dist}_T(A_1, A_2) = \inf_h \log K_h$$ "the Teichmuller distance"

Thm 1. $\mathrm{dist}_T(A_1, A_2) = \left| \log \frac{\mathrm{mod}\, A_1}{\mathrm{mod}\, A_2} \right|$

2. The optimal h_0:



$T: z \mapsto z + 2\pi i$, coverings are generated by T



covers A once

So the extremal map is linear in the log coordinates.

Topology of Riemann Surfaces

Uniformization theorem: Any simply connected Riemann Surface is conf. equiv. to one of:

1. $\hat{\mathbb{C}}$ The Riemann sphere
2. $\mathbb{C}$ The complex plane.
3. $\mathbb{D}$. The unit disk.

<u>The universal covering thm</u> If S is a R.S., there is a univ. covering

$$\begin{array}{c} M \\ \downarrow p \\ S \end{array}$$

with p holomorphic; so $M = \underset{1}{\hat{\mathbb{C}}}$ or $\underset{2}{\mathbb{C}}$ or $\underset{3}{\mathbb{D}}$.

Case 1, $\hat{\mathbb{C}}$: $\mathrm{Aut}(\hat{\mathbb{C}}) = \text{Möb} = \left\{\frac{az+b}{cz+d}\right\}$

$$= PSL_2(\mathbb{C})$$

but every Möbius transformation has

a f.p., so no subgroup of $PSL_2(\mathbb{C})$ acts freely, so $S = \hat{\mathbb{C}}$.

Case 2, $\mathbb{C}$: $\mathrm{Aut}(\mathbb{C})$ = affine maps

$\{\lambda z + c\}$ $\lambda \neq 0$.

(if $\lambda \neq 1$, has f.p.)

so deck trans must be translations

$\Rightarrow M = \mathbb{T}^2$ or non-compact $S^1 \times \mathbb{R}$

$\mathbb{T}^2 = \mathbb{C}/\text{lattice}$, $S^1 \times \mathbb{R} = \mathbb{C} \setminus \{0\}$