Yampolsky's Class, Mon Jan 19

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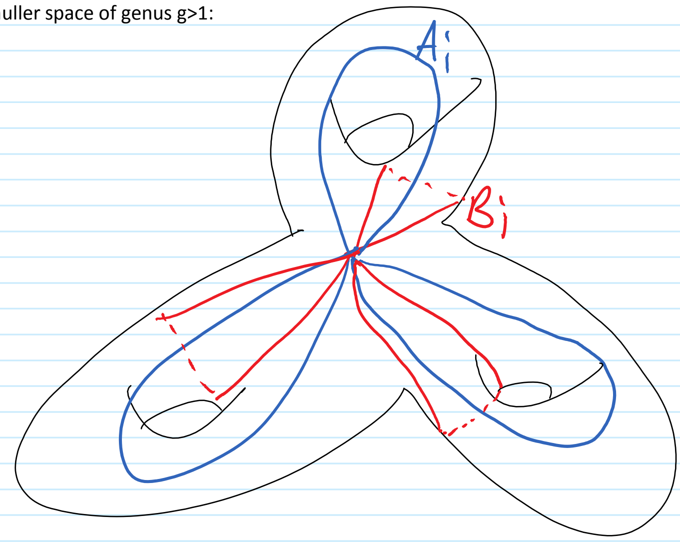
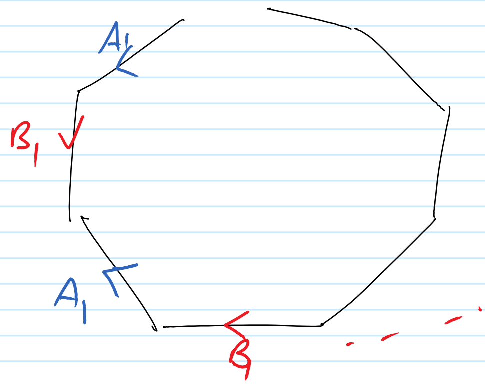
Teichmuller space of genus g>1:





Teich(S) = classes of conformal equiv. classes of marked Riemann surfaces

$S_1 \xrightarrow[h]{\sim} S_2$ dist(S_1, S_2) = inf{log K_h : $h: S_1 \to S_2 \in \text{diffeo}^+$ preserves marking}

$S = \mathbb{H}/\Gamma$, $\Gamma = \langle a_1, b_1, a_2, b_2, \dots \rangle \subset PSL_2(\mathbb{R})$

each a_i, b_i is of the form

$\gamma = \frac{az+b}{cz+d}$ s.t. $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

$6g$ parameters, mod conjugations,

$6g-3$ parameters. But there's $\prod [a_i, b_i] = 1$, so $6g-6$ parameters overall.

Equivalent definition: Fix a Riemann surface R of genus g. We consider pairs (S,f) where S is a R.S. of genus g and $f: R \xrightarrow[\text{diffeo}]{} S$ mod the following equiv:

$(S,f) \sim (S',h) \iff h \circ f^{-1}: S \to S'$ is homotopic to a bi-holomorphic map:

$$\begin{tikzcd} & S \arrow[d, dashed, "\text{conformal}"] \\ R \arrow[ur, "f"] \arrow[r, "h"'] & S' \end{tikzcd}$$

Reminder: Abs. cont. functions have "derivatives".

$f:[a,b]\to\mathbb{R}$ is abs. cont $\Rightarrow \exists g$ s.t. for a.e. $x\in[a,b]$

$$f(x)-f(a)=\int_a^x g(t)\,dt.$$

Quasi-conformal maps

Def'n $\Omega\subset\mathbb{C}$ domain. $f:\Omega\to\mathbb{C}$ is "ACL" (Abs. cont. on lines)

if f is abs. cont. on almost all horiz. & vert. lines.

Def $f:\Omega\to\mathbb{C}$ is K-quasi-conformal (K-qc)

if
- $f:\Omega\to f(\Omega)$ is homeo
- f is ACL
- $|f_{\bar z}|\le k|f_z|$ where $k=\frac{K-1}{K+1}$

* f is ACL
* $\underbrace{|f_{\bar z}| \leq k|f_z|}_{\text{in an a.e. sense.}}$ where $k = \frac{K-1}{K+1}$

Claim f is K-qc $\Rightarrow f_z, f_{\bar z} \in L^2_{loc}$

Thm Suppose f is a homeo, f has distributional particles then f is ACL

So Alt. Def: f is K-qc if

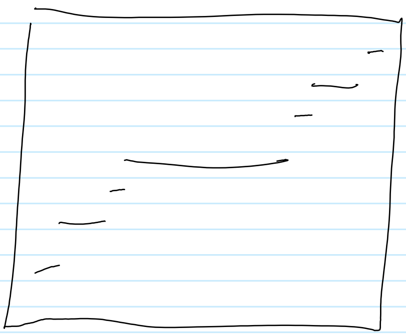
* f is homeo$^+$
* f_x, f_y exist as distributions & are locally integrable.
* $|f_{\bar z}| \leq k|f_z|$ where $k = \frac{K-1}{K+1}$

Thm (in Ahlfors) Suppose $f \in$ homeo$^+$, ACL, $f_{\bar z} = 0$ a.e. Then f is conformal.

Everybody's favorite almost-counter example:

The Devil's staircase:

A surjective $\eta: [0,1] \to [0,1]$ flat on the cantor set C.



Set

$$f(x,y) = (x, y + \eta(x))$$

* homeo
* diffable a.e., w/ derivatives a.e. $= 1$.
  So $f_{\bar z} = 0$ a.e.

$\longrightarrow$ Derivatives are not dist. derivatives.

Thm If f is qc then area is an

absolutely cont. function of the set being measured.

* $$f(A) = \iint_A \mathrm{Jac}(f)$$

* So if $\mathrm{area}(A) = 0$ then $\mathrm{area}(f(A)) = 0$.

Extremal Length: Let Γ be a family of locally rectifiable curves in $\mathbb{C}$.

Def'n $\rho : \mathbb{C} \to \mathbb{R}$ is "admissable" if

* ρ is measurable.
* $\rho \geq 0$
* $A(\rho) = \int_{\mathbb{C}} \rho^2 \, dx\, dy \neq 0, \infty$

Let $\gamma \in \Gamma$. Set

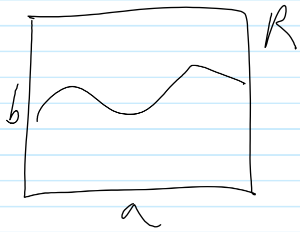
$$L_\gamma(\rho) = \begin{cases} \int_\gamma \rho |dz| & \text{if defined} \\ \infty & \text{otherwise} \end{cases}$$

Set $L(\rho) = \inf_{\gamma \in \Gamma} L_\gamma(\rho)$.

Def'n The extremal length of Γ,

$$\lambda(\Gamma) = \sup_\rho L(\rho)^2 / A(\rho)$$

Example 1



$\Gamma = \{$ all curves connecting left & right inside $R \}$

prop $\lambda(\Gamma) = \dfrac{a}{b}$.

proof $\lambda(\Gamma) \leq \frac{a}{b}$: Take any ρ

$L(\rho) \leq \int_0^a \rho(x+iy)\,dx$ so

$bL(\rho) \leq \iint \rho(x+iy)\,dx\,dy$ so by C-S:

$b^2 L(\rho)^2 \leq ab \iint \rho^2\,dx\,dy \leq ab A(\rho)$

For the other direction, pick $\rho = 1$ on R, 0 outside.

Comment: "$\Gamma_1 < \Gamma_2$" if $\forall \gamma_2 \in \Gamma_2$ contains a curve in Γ_1 as a subcurve.

claim $\Gamma_1 < \Gamma_2 \Rightarrow \lambda(\Gamma_1) \leq \lambda(\Gamma_2)$

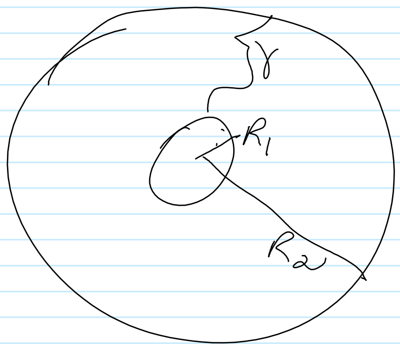
Thm Suppose Γ_1 & Γ_2 are contained in non-intersecting open sets. Then

* $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$
* Declare $\Gamma_1 + \Gamma_2 :=$ all ways of concatenating a curve in γ_1 w/ a curve in γ_2 [curves need not be continuous!]

Then

$$\lambda(\Gamma_1 + \Gamma_2) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$$

Example 2



Γ curves connecting inner & outer bndry.

Prop $\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{R_2}{R_1}$

Pf $\int_{R_1}^{R_2} \rho\, dr \geq L(\rho) \quad \Rightarrow \quad \iint \rho\, dr\, d\theta \geq 2\pi L(\rho)$

Use C–S.... Equality is achieved when $\rho = \frac{1}{r}$.

Thm Let f be K-qc, $\Gamma' = f(\Gamma)$. Then

$$\frac{\lambda(\Gamma)}{K} \leq \lambda(\Gamma') \leq K\lambda(\Gamma) \quad \left(\begin{matrix}\text{proof}\\ \text{later}\end{matrix}\right)$$

Corollary $\lambda(\Gamma)$ is a conformal invariant.