Peter Samuelson: Introduction to Macdonald Polynomials

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Ref: Macdonald's original paper, "a new class of symmetric functions", 1988.

Notation: $\Lambda_n := \mathbb{Z}[x_1 \dots x_n]^{S_n}$

$\Lambda_{n+1} \to \Lambda_n$ by $x_{n+1} \to 0$

$\Lambda = \varprojlim \Lambda_n$ "symmetric functions"

can be viewed within $\mathbb{Z}[[x_1, \dots]]^{S_\infty}$

<u>Def</u> A partition is $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots)$

s.t. $|\lambda| := \sum \lambda_i < \infty$. Partially ordered by

$\lambda \geq \mu \iff |\lambda| = |\mu|$ & $\forall r$ $\lambda_1 + \dots + \lambda_r \geq \mu_1 + \dots + \mu_r$

<u>Examples</u>

$m_\lambda := S_\infty \cdot (x_1^{\lambda_1} x_2^{\lambda_2} \dots) \Big/ \#\sigma \in S_n,\ \sigma\lambda = \lambda$

e.g. $m_{(2,1)} = \sum_{i \neq j} x_i^2 x_j$

Power sums: $p_r := m_{(r)} = \sum_i x_i^r$

$p_\lambda := p_{\lambda_1} \cdot p_{\lambda_2} \cdot p_{\lambda_3} \dots$

Both m_λ & p_λ are $\mathbb{Z}$-bases of Λ.

Schur functions:

$$D_\lambda = \det\left(x_i^{\lambda_j + n - j}\right) \in \Lambda_n$$

$$D_0 = \prod_{i \neq j} (x_i - x_j)$$

$$s_\lambda := D_\lambda / D_0$$

satisfy $s_\lambda(x_1 \ldots x_n, 0) = s_\lambda(x_1 \ldots x_n)$,

so make sense on Λ.

$\langle -, - \rangle : \Lambda \otimes \Lambda \to \mathbb{Z}$ by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \qquad z_\lambda = \prod (r^{m_r} m_r!)$$
$$\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots)$$

Lemma: The s_λ are determined by

A. $s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu \qquad K_{\lambda\mu} \in \mathbb{Z}$

Kostka numbers.

B. $\langle s_\lambda, s_\mu \rangle = 0$ if $\lambda \neq \mu$

Property: $K_{\lambda\mu} \geq 0$

One (rough) explanation is that

s_λ are characters of GL_n irreps
& m_λ are "orbital sums" under S_n.

.... the use rep theory.

Gram-Schmidt! ....

Macdonald poly's: Switch coeffs to

$$F := \mathbb{Q}(q,t)$$

$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda(q,t)$ where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

Thm (Macdonald) $\exists!$ basis $P_\lambda(q,t)$
satisfying (A) & (B) for this pairing.

Some specializations:

1. $P_\lambda(q,q) = s_\lambda$

2. $P_\lambda(0,t) =$ Hall-Littlewood functions
(come from Abelian p-groups).

3. $q = t^\alpha$, $\lim_{t\to 1} P_\lambda(t^\alpha, t)$ "Jack symm. fns"

4 $P_\lambda(q,1) = m_\lambda$

$P_\lambda(1,t) = e_\lambda$ elementary symm. functions