# Lee@Colloq: Positivity for cluster algebras

January-14-15 4:08 PM

Easy-looking open problem:

$$a_0 = a_1 = 1 \qquad a_{n+1} = \frac{a_n^3 + 1}{a_{n-1}}$$

1, 1, 2, 9, 365, 5403014, ...

Are these all integers by using number theory only?

Overview A cluster algebra is an algebra generated by distinguished generators "cluster variables" which are constructed by recursive relations "mutations".

highly non-trivial

Thm (Fomin-Zelevinsky, 2001) The cluster variables are Laurent polynomials with integer coefficients.

Conjecture These coeffs are all positive.

Thm (L-Schiffler, 2013-14) Conjecture is true.

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Mutation — Introduced by Seiberg 1994

Cluster algebras:* by F-Z 2001.

*related to Lie algs / alg comb / rep theory / alg geom / num. theory / hyp. geom / symp. geom / Poisson geom / low dim top /

/ Knot theory / Teich. theory / triangulations of surfaces / / Schrödinger eq / String theory / KP-solitons.

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Rank 2 Cluster Algebras $(\bullet \xrightarrow{r} \bullet) = Q$

quiver (directed graph)

rank = # of vertices in the quiver.

Let $x_1, x_2$ be indeterminants. Define

$$x_{n+1} = \frac{x_n^r + 1}{x_{n-1}} \qquad \text{"mutation"}$$

Def $x_n$ are "cluster variables"

Def The cluster assoc. to $Q$ is the subalgebra of $\mathbb{Q}(x_1, x_2)$ gen. by the cluster variables.

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at $r=1$

$$x_3 = \frac{x_2+1}{x_1} \qquad x_4 = \frac{x_1+x_2+1}{x_1 x_2}$$

$$x_5 = \ldots = \frac{x_1+1}{x_2} \quad - - - -$$

Thm (FZ, 2001) The $x_n$'s are Laurent poly in $x_1, x_2$

Thm (Nakajima, 2009) at $r=1$, coeffs are all positive.

Thm (L-Schiffler, 2011) Comb. formula for coeffs.

# KP Soliton (Following Kodama-Williams)

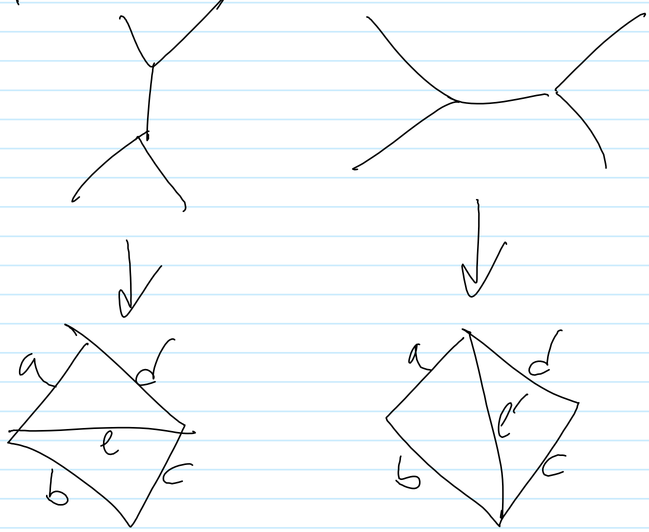
Consider $n$ waves on surface of shallow water

1. Solve the corresponding KP eqn's.
2. Take a limit on the sol'n set.
3. There are $\underbrace{C_n}_{\text{catalan}}$-many limit behavior

$$\underset{\text{catalan}}{C_n} = \frac{1}{n+1}\binom{2n}{n} = \#\text{ of triangulations of } (n+2)\text{-gon.}$$

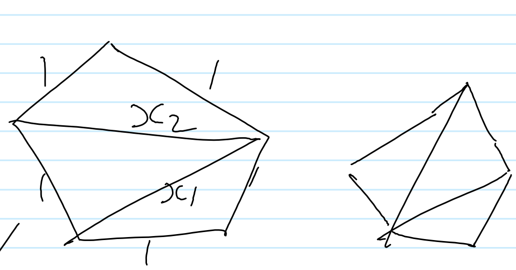
4. Look at dual graphs.

Example: $n=2$

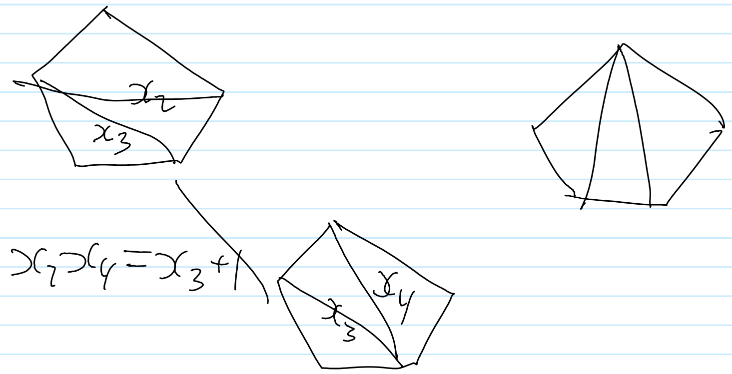



Then $ee' = ac + bd$ "skein relation"

## $n=3$




$x_1 x_3 = x_2 + 1$




## More general cluster algebra

Seiberg duality (Quiver mutations)

locally looks like

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] \arrow[d, "\mu_2"] & 3 \\
1 \arrow[rr, bend right] & 2 \arrow[l] & 3 \arrow[l]
\end{tikzcd}$$

Def (seed mutation)

$$\Big(Q, \underbrace{(x_1 \dots x_n)}_{\text{cluster}}\Big) \xrightarrow{\mu_k} \Big(\mu_k(Q), \underbrace{x_1 \dots x_k' \; x_n}_{\text{new cluster}}\Big)$$

$$x_k' = \frac{\prod_{i \to k} x_i + \prod_{j \leftarrow k} x_j}{x_k}$$

**Def** Elements of clusters are called cluster variables

**Thm** (F-Z) Laurent.

Thm (LS) positivity

Thm (Gross-Hacking-Keel-Kontsevich)

positivity in a more general context.