Kapovitch's class, Mon Jan 5: Organizational, statement of the h-cobordism theorem, basic Morse theory

January-05-15 11:11 AM

MAT1318HS
SEMINAR IN GEOMETRY AND TOPOLOGY
V. Kapovitch
(View Timetable)

MWF 11

The course is intended to provide a short introduction to several important areas of geometry and topology.

Some of the topics that will be covered are:

1. Geometrization conjecture of 3-manifolds. Geometrization is a program of classifying closed 3-manifolds by decomposing them into a number of "geometric" pieces. We will talk about Thurston's geometrization theorem for Haken manifolds and orbifolds, Hamilton's Ricci flow approach, Perelmans's solution of the general geometrization problem.
2. h-cobordism theorem and applications. h-cobordism theorem underlies the theory of smooth manifolds in high dimensions. In particular, it implies the Poincare conjecture - that any smooth manifold of dimension at least 5 homotopy equivalent to a sphere is homeomorphic to a sphere. We will discuss Morse theory, h-cobordism, Poincare conjecture, classification of manifolds.
3. Rational homotopy theory. Effectively computing full homotopy invariants of a space is a hard problem. In particular, homotopy groups of S^2 are not known. It turns out that the problem becomes much easier rationally and the entire rational homotopy type of a space can be encoded in an effectively computable algebraic model. We will discuss rational homotopy equivalences, minimal Sullivan models, formal spaces and elliptic spaces.
4. Geometric group theory. Geometric group theory is the study of the large scale geometry of a group. Many problems in combinatorial group theory such as the isomorphism problem, the word problem and the conjugacy problem have been shown to be unsolvable. However, adding some geometric conditions on the group, e.g. assuming that the group is Gromov-hyperbolic, will guarantee the existence of algorithms.

Prerequisites:
Knowledge of basic differential topology, algebraic topology and differential geometry.

From <http://www.math.toronto.edu/cms/2014-2015-proposed-graduate-courses-descriptions/#MAT1318HS>

1. h-cobordism.

Morse theory, h-cobordism, Whitney's trick, Poincare's conjecture, maybe exotic spheres, Surgery.

Sources: Milnor's book on Morse Theory
Milnor's "Lectures on h-cobordism"
Ranicki "Algebraic and geometric surgery".

2. Rational homotopy theory.

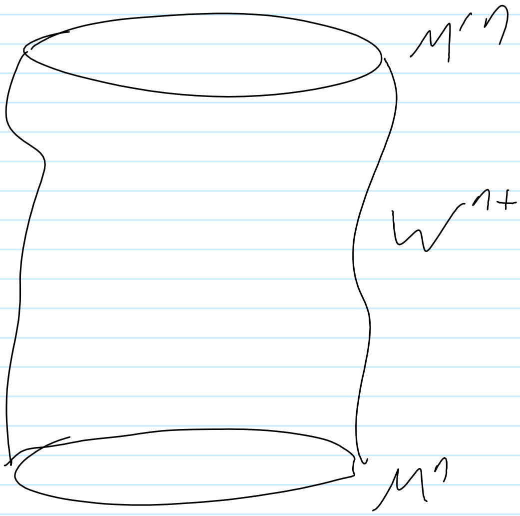
Source:

3. Geometrization.

$$\frac{dg}{dt} = -Ric_{g_t}$$

4. Geometric group theory.

The h-cobordism theorem



$\partial W = M \sqcup M'$

Suppose

$M \hookrightarrow W$

and

$M' \hookrightarrow W$

are homotopy equivalences,

Thm (Smale) if $n \geq 5$ and $\pi_1(W) = 0$ then W is a product. (And so $M \cong M'$)

Classification of manifolds:

Find complete diffeomorphism invariants of manifolds.

Fails because even isomorphisms of π_1 can be decided.

Surgery theory Let $f: M_1 \to M_2$ be a homotopy equivalence. Is it homotopic to a diffeomorphism?

What do you do if no f is given in advance?

Example $T^1 S^6$ & $G_2/SU(3)$ are homeomorphic but not diffeomorphic.

Morse theory: $f: M^n \to \mathbb{R}$ is "Morse" if

Morse theory: $f: M^n \to \mathbb{R}$ is "Morse" if all critical points of f are non-degenerate:

$\forall p$ $df_p = 0 \Longrightarrow$ the Hessian (a symmetric bilinear form on T_pM) is non-degenerate

$$\mathrm{Hess}_p(f) = \frac{\partial f}{\partial x_i \partial x_j}(p) \quad \left(\begin{array}{l}\text{well defined}\\ \text{if } df_p = 0\end{array}\right)$$

The fundamental lemma of Morse theory:

If $f: M \to \mathbb{R}$ is Morse and p is critical, then in some coordinates near p,

$$f = -x_1^2 - x_2^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2$$