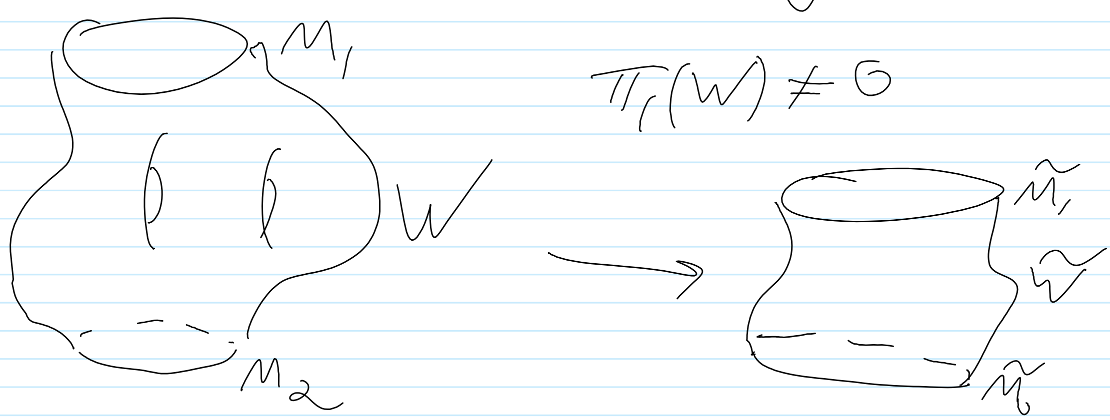
Kapovitch's class, Fri Jan 16: The non-simply-connected case

January-16-15 11:08 AM

What happens in the non-simply connected case?

$\pi_1(W) \neq 0$



Whitney's trick used $\pi_1 = 0$, so no longer holds.

Lift to a universal cover....

$C_*(\tilde{W}, \tilde{M}_0)$ is a free $\mathbb{Z}\pi_1$ module...

Intersections are elements of $\mathbb{Z}\pi_1$....

∂ matrices can no longer be brought to the form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$

If $A, B \in GL(n, \mathbb{Z})$ then you can get from A to B by elementary row operations.

In $GL(n, \mathbb{Z}\pi_1)$ the failure of the above fact is measured by the Whitehead group of π_1 $Wh(\pi_1)$

$\Longrightarrow$ If $f: X - X$ is an h-cobordism with $\pi_1 \neq 0$, get $W(f) \in Wh(\pi_1(X))$

if $W(f) = 0 \Rightarrow f$ is called "simple homotopy equiv"

In that case f is called an "s-cobordism"

<u>Thm</u> If $n \geq 5$ any s-cobordism is a product.

$Wh(\{1\}) = 1 \qquad Wh(\mathbb{Z}) = 1$

precise def of Wh:

$$GL(n, \mathbb{Z}\pi) \longrightarrow GL(n+1, \mathbb{Z}\pi) \to \dots$$

Let $GL(\mathbb{Z}\pi)$ by the direct limit in $GL(\mathbb{Z}\pi)$ we have elementary matrices.

$E \subset GL(\mathbb{Z}\pi)$ is the subgroup generated by elem. matrices.

<u>Fact</u> $E \triangleleft GL(\mathbb{Z}\pi)$, $E = [GL, GL]$

$$GL/E \Big/ \underset{g \in \pi}{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & \pm g, 1 \end{pmatrix}} =: Wh(\pi)$$

Classification of manifolds $(n \geq 5)$

To classify manifold up to diffeo we need to classify them up to h-cobordism.

Def'n M_1^n & M_2^n are "cobordant" if

$\exists W^{n+1}$ s.t $\partial W = \overline{M_0} \sqcup M_1$

Note $M_1 \sim N_1$, $M_2 \sim N_2 \Rightarrow M_1 \times N_1 \sim M_2 \times N_2$

Get "cobordism rings"

$\Omega_*^{O(N)}$ unoriented case

$\Omega_*^{SO(N)}$ oriented case.

A manifold is null-cobordant if it is cobordant to $\emptyset$.

Examples $\mathbb{R}P^2$ is not null-cobordant.

$\mathbb{C}P^2$ is not null-cobordant.

$\mathbb{C}P^{2n}$ is not null-cobordant.

Thm $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \dots]$

$\mathbb{C}P^1 \sim \emptyset$, $\mathbb{C}P^{2n+1} \sim \emptyset$

Proof that $\mathbb{C}P^{2n} \not\sim \emptyset$:

Thm if $M^{4n} = \partial W^{4n+1}$ then $\operatorname{sign}(M) = 0$.

But $\operatorname{sign} \mathbb{C}P^{2n} = 1$

Lemma If $\langle\,\rangle$ is a quadratic form on $\mathbb{R}^{2n}$ and $V \subset \mathbb{R}^{2n}$ is n-dim and $\langle v, v \rangle = 0$ then $\operatorname{sign}\langle\,\rangle = 0$.