Tangles, Wheels, Balloons

Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

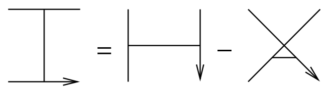
Why I like "non-commutative"? With $FA(x_i)$ the free associative non-commutative algebra,

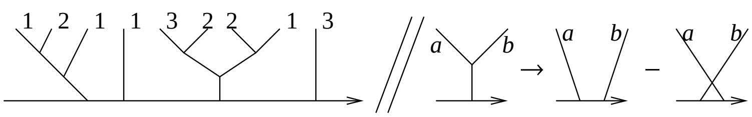
$$\dim \mathbb{Q}[x,y]_d \sim d \ll 2^d \sim \dim FA(x,y)_d.$$

Why I like "computable"?

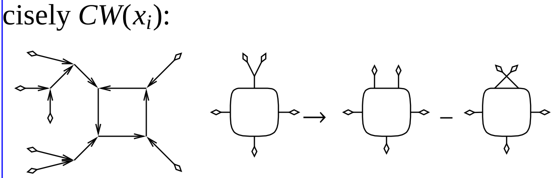
- Because I'm weird.
- Note that π_1 isn't computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i); $FL(x_i)$ = (binary trees with AS vertices and coloured leafs)/(IHX relations). There an obvious map $FA(FL(x_i)) \to FA(x_i)$ defined by $[a,b] \to ab - ba$, which in itself, is IHX.

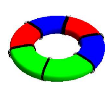




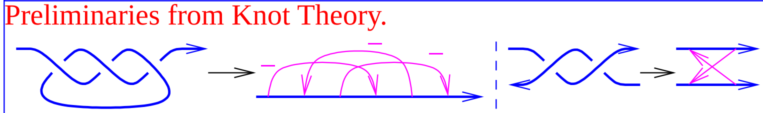
$CW(x_i)$ denotes the vector space of cyclic words in (x_i): $CW(x_i) = FA(x_i)/(x_iw = wx_i)$. There an obvious map $CW(FL(x_i)) \to CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1,\ldots,n\}$, modulo AS and IHX, is precisely $CW(x_i)$:



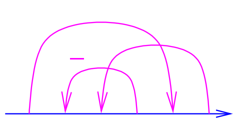
Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.



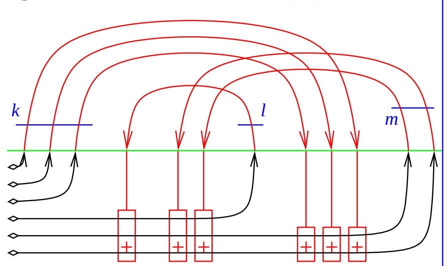
Preliminaries from Knot Theory.

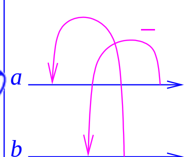


Theorem. ~~ζ~~ W, the connected part of the procedure below, is an invariant of S-component tangles with values in $CW(S)$:

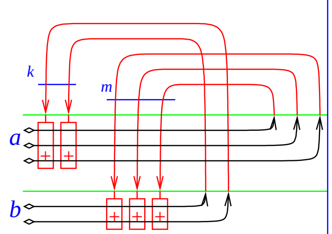


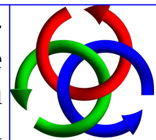
$$\xrightarrow{e^{W}} \sum_{k,l,m\geq 0} \frac{(+)^k(-)^l(+)^m}{k!l!m!}$$



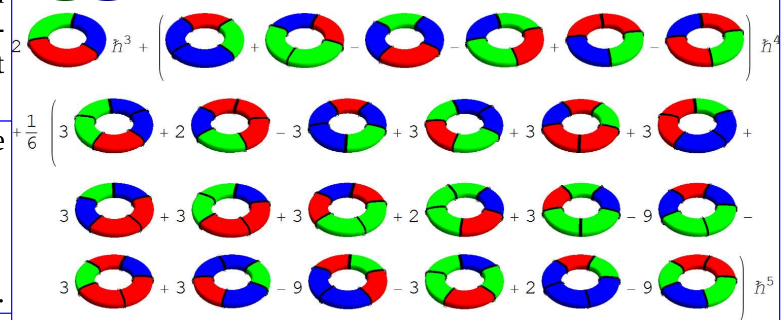


$$\xrightarrow{e^{W}} \sum_{k,m\geq 0} \frac{(+)^k(-)^m}{k!m!}$$



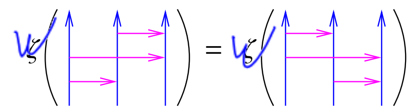


~~ζ~~ W is practically computable! For the Borromean tangle, to degree 5, the result is: (see [BN])

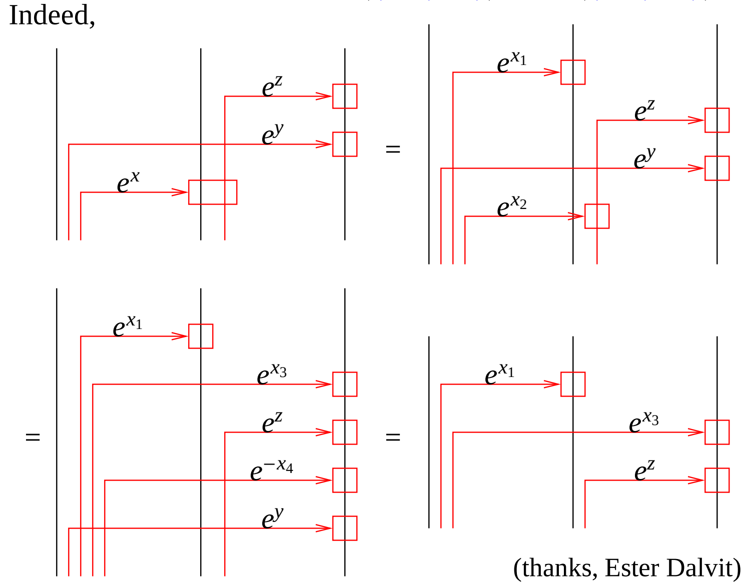


Proof of Invariance.

Need to show:



Indeed,



(thanks, Ester Dalvit)

Further Facts:

* W is really the 2nd part of a (trees, wheels)-valued invariant $\zeta = (\lambda, W)$. The tree part λ is just a repackaging of the Milnor μ-invariants.
* On u-tangles, ζ is equivalent to the trees & wheels part of the Kontsevich integral, except it is defined with no need for a choice of parenthetisation.
* on long u-knots / round u-knots, W is equivalent to the Alexander polynomial.
* The multivariable Alexander polynomial (and Levine's factorization thereof) is contained in the Abelianization of ζ
* W vanishes on braids.
* ~~Further~~
* should be summed and categorified.

Extends to V and descends to W:

Meaning, satisfies ... also satisfies

Agrees w/ BN-Dancso [ ] & [BN]

Import a section on simply-knotted 2-knots

From HUJI-140101

• Agrees with BN-Dancso [BND1, BND2] and with [BN]. • ~~In-practice computable?~~ • ~~Vanishes on braids.~~ • ~~Extends to w.~~ • ~~Contains Alexander.~~ • ~~The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA).~~ • ~~Related to / extends Farber's [Fa]?~~ • ~~Should be summed and categorified.~~

Import a section about BF

From GoodFormulas

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BND1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial,* http://drorbn.net/AcademicPensieve/Projects/WKO1, arXiv:1405.1956.

[BND2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem,* http://drorbn.net/AcademicPensieve/Projects/WKO2, arXiv:1405.1955.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants,* Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links,* Math. Ann. **293** (1992) 543–568.

[Le] J. Levine, *A Factorization of the Conway Polynomial,* Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology,* Alg. and Geom. Top. **7** (2007) 47-92, arXiv:math/0609742.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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