

Attempt a \mathcal{D}_3 ? ^{too} _{tough}

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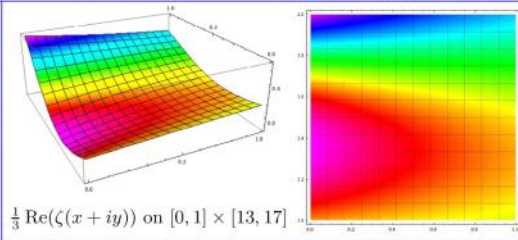
Dessert: Hilbert's 13th Problem, in Full Colour

Dror Bar-Natan, Toronto November 2014. More at <http://www.math.toronto.edu/~drorbn/Talks/Fields-1411>

activate

Abstract. To break a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnold solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.

In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnold showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that **any** continuous function f of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function “+” (addition). For $f(x, y) = xy$, this may be $xy = \exp(\log x + \log y)$. For $f(x, y, z) = x^y/z$, this may be $\exp(\exp(\log y + \log \log x) + (-\log z))$. What might it be for (say) the real part of the Riemann zeta function?



$\frac{1}{3} \operatorname{Re}(\zeta(x + iy))$ on $[0, 1] \times [13, 17]$
 Fix an irrational $\lambda > 0$, say $\lambda = (\sqrt{5} - 1)/2$. All functions are continuous.

Theorem. There exist five $\phi_i : [0, 1] \rightarrow [0, 1]$ ($1 \leq i \leq 5$) so that for every $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ there exists a $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that

$$f(x, y) = \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))$$

for every $x, y \in [0, 1]$.



Hilbert



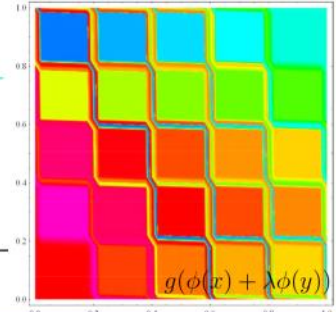
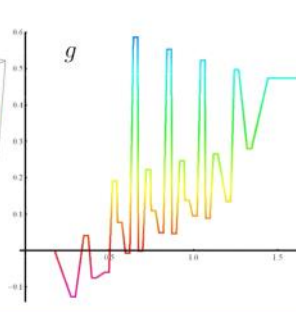
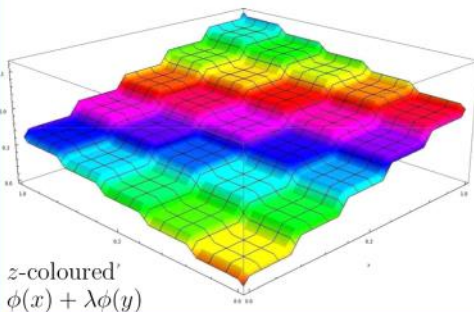
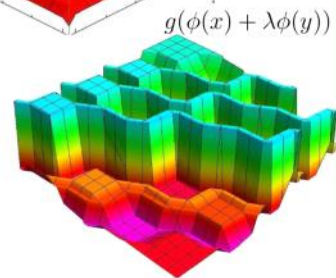
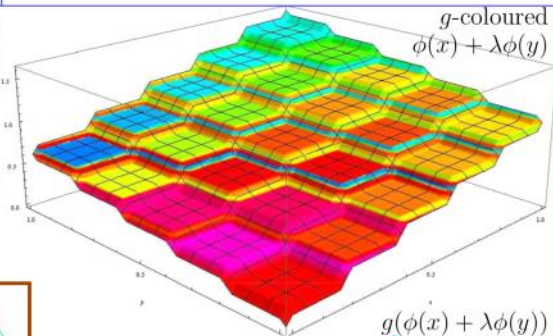
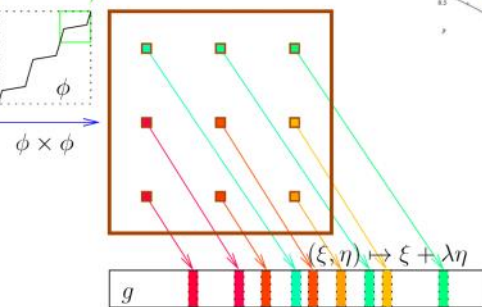
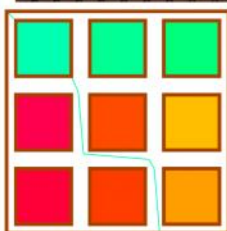
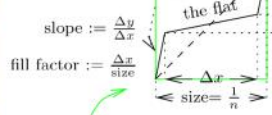
Kolmogorov



Arnold (by Moser)

Step 1. If $\epsilon > 0$ and $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, then there exists $\phi : [0, 1] \rightarrow [0, 1]$ and $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that $|f(x, y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on at least 98% of the area of $[0, 1] \times [0, 1]$.

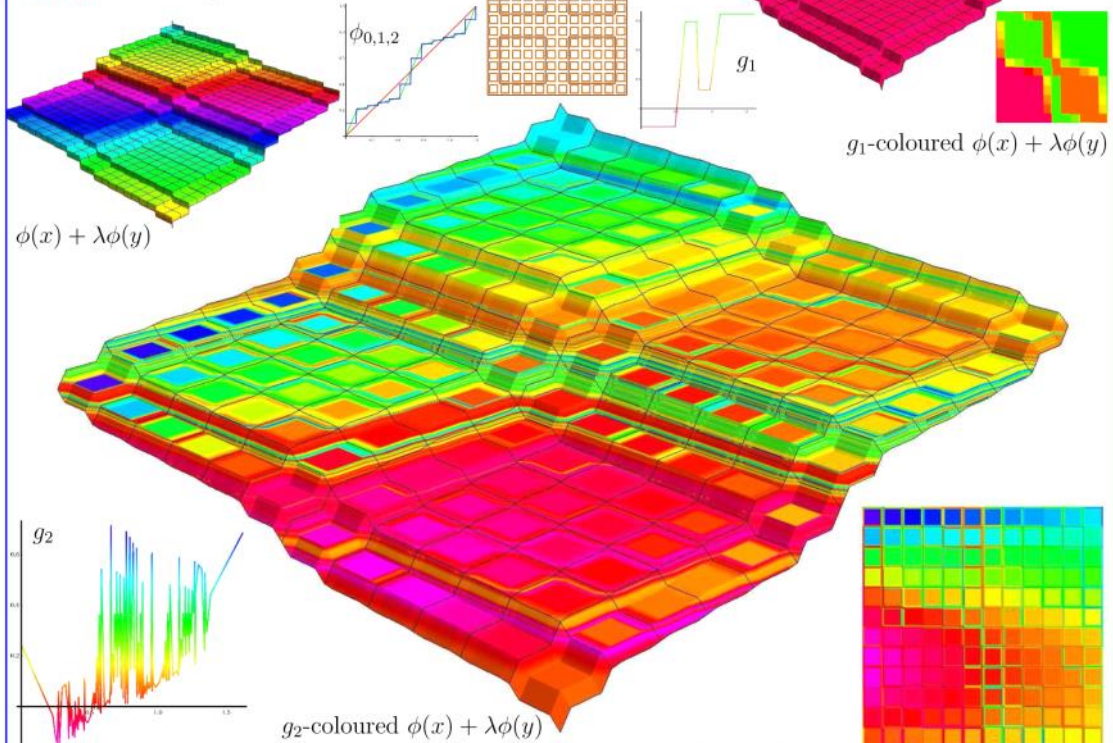
The key. “Poorify” chocolate bars.



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Step 2. There exists $\phi : [0, 1] \rightarrow [0, 1]$ so that for every $\epsilon > 0$ and every $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ there exists a $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that $|f(x, y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0, 1] \times [0, 1]$.

The key. "Iterated poorification".



Step 3. There exist $\phi_i : [0, 1] \rightarrow [0, 1]$ ($1 \leq i \leq 5$) so that for every $\epsilon > 0$ and every $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ there exists a $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that

$$|f(x, y) - \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) \|f\|_\infty$$

for every $x, y \in [0, 1]$.

The key. "Shift the chocolates"...



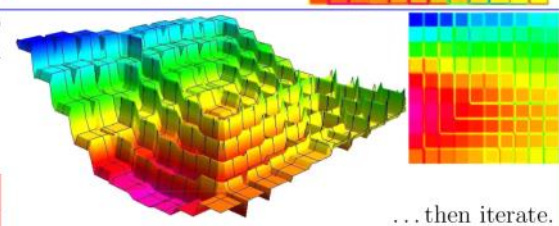
Step 4. We are done.

The key. Learn from the artillery!

Set $Tg := \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))$, $f_1 := f$, $M := \|f\|$, and iterate "shooting and adjusting". Find g_1 with $\|g_1\| \leq M$ and $\|f_2 := f_1 - Tg_1\| \leq \frac{3}{4}M$. Find g_2 with $\|g_2\| \leq \frac{3}{4}M$ and $\|f_3 := f_2 - Tg_2\| \leq (\frac{3}{4})^2M$. Find g_3 with $\|g_3\| \leq (\frac{3}{4})^2M$ and $\|f_4 := f_3 - Tg_3\| \leq (\frac{3}{4})^3M$. Continue to eternity. When done, set $g = \sum g_k$ and note that $f = Tg$ as required.

Exercise 1. Do the m -dimensional case.

Exercise 2. Do \mathbb{R}^m instead of just I^m .



... then iterate.

Bad Mousing. I think low of PowerPoint!

- No good feelings towards Impress-Beamer, etc., either.
- Must sync with speaker, can't look back at key points, don't know what to look forward too.
- No upper bound to useless content.
- Nothing to take home.

Propaganda. I love handouts!

- I have nothing to hide and you can take what you want, forwards, backwards, here and at home.
- What doesn't fit on one sheet can't be done in one hour.
- It costs many hours and a few pennies. The audience's worth it!
- Can put hyperbolic geometry in every talk!



or There's real math in the handout itself!

how best go?

how best go? ✓

Nice ref (from Mike Shub): A.G. Vitushkin, "On Representations of Functions by Means of Superpositions and Related Topics". Nah.

watch old video?