Pensieve header: qΓ-Calculus.

```
SetDirectory["C:/drorbn/AcademicPensieve/2014-11/"]
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C:\drorbn\AcademicPensieve\2014-11

```
<< KnotTheory`
```

Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.

```
ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ ⧴ a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
```

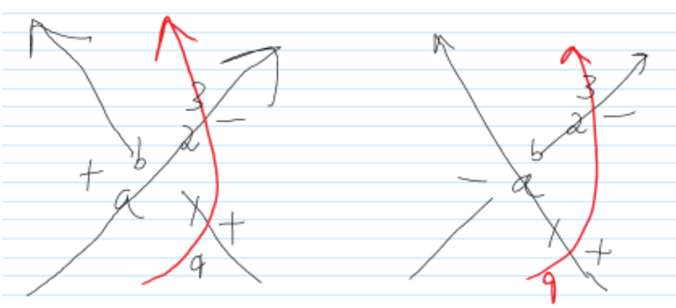
$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a,h_a}\lambda & \partial_{t_a,h_b}\lambda & \partial_{t_a}\lambda \\ \partial_{t_b,h_a}\lambda & \partial_{t_b,h_b}\lambda & \partial_{t_b}\lambda \\ \partial_{h_a}\lambda & \partial_{h_b}\lambda & \lambda \end{pmatrix} \text{ /. } (t \mid h)_{a|b} \to 0;$$

$$\Gamma\left[(\mu = 1 - \beta)\ \omega,\ \{t_c, 1\}.\begin{pmatrix} \gamma + \alpha\,\delta / \mu & \epsilon + \delta\,\theta / \mu \\ \phi + \alpha\,\psi / \mu & \Xi + \psi\,\theta / \mu \end{pmatrix}.\{h_c, 1\}\right]$$

```
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
```

$$Rp_{a\_b\_} := \Gamma\left[1,\ \{t_a, t_b\}.\begin{pmatrix} 1 & 1 - T_a \\ 0 & T_a \end{pmatrix}.\{h_a, h_b\}\right];$$

```
Rm_{a_b_} := Rp_{ab} /. T_a → 1 / T_a;
```



```
{Rp_{a,b} Rp_{q,1} Rm_{3,2} // m_{a,2→a} // m_{1,b→b} // m_{q,3→q}} /. T_q → q
```

$$\left\{\begin{pmatrix} 1 & h_a & h_b & h_q \\ t_a & \frac{1}{q} & 1 - T_a & 0 \\ t_b & 0 & q\,T_a & 0 \\ t_q & \frac{-1+q}{q} & -(-1+q)\,T_a & 1 \end{pmatrix}\right\}$$

$$\gamma = \Gamma\left[\omega,\ (t_1\ \ t_2\ \ t_3\ \ t_S).\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\begin{pmatrix}h_1\\ h_2\\ h_3\\ h_S\end{pmatrix}\ \text{// Total // Total}\right]$$

$$\begin{pmatrix}\omega & h_1 & h_2 & h_3 & h_S\\ t_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ t_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ t_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ t_S & \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}$$

γ // $m_{12\to 1}$ // $m_{13\to 1}$

$$\left(\begin{matrix}\omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & h_1\\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}\\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}\end{matrix}\right.$$

γ // $m_{23\to 2}$ // $m_{12\to 1}$

$$\left(\begin{matrix}\omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & h_1\\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}\\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}\cdots}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}\end{matrix}\right.$$

MetaAssoc

$$\gamma = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_S\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right];$$

(γ // $m_{12\to 1}$ // $m_{13\to 1}$) == (γ // $m_{23\to 2}$ // $m_{12\to 1}$)

MetaAssoc

True

R3

$\{Rm_{51}\,Rm_{62}\,Rp_{34}$ // $m_{14\to 1}$ // $m_{25\to 2}$ // $m_{36\to 3}$,
$Rp_{61}\,Rm_{24}\,Rm_{35}$ // $m_{14\to 1}$ // $m_{25\to 2}$ // $m_{36\to 3}\}$

R3

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$

$\gamma = \mathrm{Rm}_{12,1}\, \mathrm{Rm}_{27}\, \mathrm{Rm}_{83}\, \mathrm{Rm}_{4,11}\, \mathrm{Rp}_{16,5}\, \mathrm{Rp}_{6,13}\, \mathrm{Rp}_{14,9}\, \mathrm{Rp}_{10,15}$

1	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{1}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2	0	1	0	0	0	0	$\frac{-1+T_2}{T_2}$	0	0	0	0	0	0	0	0	0
t_3	0	0	$\frac{1}{T_8}$	0	0	0	0	0	0	0	0	0	0	0	0	0
t_4	0	0	0	1	0	0	0	0	0	0	$\frac{-1+T_4}{T_4}$	0	0	0	0	0
t_5	0	0	0	0	T_{16}	0	0	0	0	0	0	0	0	0	0	0
t_6	0	0	0	0	0	1	0	0	0	0	0	0	$1-T_6$	0	0	0
t_7	0	0	0	0	0	0	$\frac{1}{T_2}$	0	0	0	0	0	0	0	0	0
t_8	0	0	$\frac{-1+T_8}{T_8}$	0	0	0	0	1	0	0	0	0	0	0	0	0
t_9	0	0	0	0	0	0	0	0	T_{14}	0	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$1-T_{10}$	0
t_{11}	0	0	0	0	0	0	0	0	0	0	$\frac{1}{T_4}$	0	0	0	0	0
t_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
t_{13}	0	0	0	0	0	0	0	0	0	0	0	0	T_6	0	0	0
t_{14}	0	0	0	0	0	0	0	0	$1-T_{14}$	0	0	0	0	1	0	0
t_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{10}	0
t_{16}	0	0	0	0	$1-T_{16}$	0	0	0	0	0	0	0	0	0	0	1

Do[γ = γ // $m_{1k\to1}$, {k, 2, 10}]; γ

$\frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2}$	h_1	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}	h_{16}
t_1	$\frac{T_{14}\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14} T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	0	$-\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14}}{T_1^2+T_{16}-T_1 T_{16}}$	0	$1-T_1$	0
t_{11}	0	$\frac{1}{T_1}$	0	0	0	0	0
t_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	1	0	0	0	0
t_{13}	0	0	0	T_1	0	0	0
t_{14}	$-\frac{(-1+T_{14})\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$-\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14}) T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	0	$\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}}$	1	0	0
t_{15}	0	0	0	0	0	T_1	0
t_{16}	$-\frac{T_1(-1+T_{16})}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)}$	$-\frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	$\frac{(-1+T_1)^2(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	0	1

8_17

$\gamma = \mathrm{Rm}_{12,1}\, \mathrm{Rm}_{27}\, \mathrm{Rm}_{83}\, \mathrm{Rm}_{4,11}\, \mathrm{Rp}_{16,5}\, \mathrm{Rp}_{6,13}\, \mathrm{Rp}_{14,9}\, \mathrm{Rp}_{10,15}$;
Do[γ = γ // $m_{1k\to1}$, {k, 2, 16}]; γ

8_17

$-\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3}$	h_1
t_1	1

$\gamma = \mathbf{Rm}_{16}\,\mathbf{Rp}_{74}\,\mathbf{Rp}_{52}\,\mathbf{Rm}_{8,11}\,\mathbf{Rp}_{12,9}\,\mathbf{Rp}_{10,3}$

$$\begin{pmatrix}
1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\
t_1 & 1 & 0 & 0 & 0 & 0 & \frac{-1+T_1}{T_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
t_2 & 0 & T_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_3 & 0 & 0 & T_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_4 & 0 & 0 & 0 & T_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_5 & 0 & 1-T_5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_6 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
t_7 & 0 & 0 & 0 & 1-T_7 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
t_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{-1+T_8}{T_8} & 0 \\
t_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{12} & 0 & 0 & 0 \\
t_{10} & 0 & 0 & 1-T_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
t_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_8} & 0 \\
t_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-T_{12} & 0 & 0 & 1
\end{pmatrix}$$

$\gamma \,//\, \mathbf{m}_{14\to1} \,//\, \mathbf{m}_{15\to1} \,//\, \mathbf{m}_{16\to1} \,//\, \mathbf{m}_{27\to2} \,//\, \mathbf{m}_{28\to2} \,//\, \mathbf{m}_{29\to2} \,//\, \mathbf{m}_{2,10\to2} \,//\, \mathbf{m}_{2,11\to2} \,//\, \mathbf{m}_{3,12\to3}$

$$\begin{pmatrix}
\frac{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)}{T_1 T_2} & h_1 & h_2 & h_3 \\
t_1 & -\frac{T_2}{-T_1-T_2+T_1 T_2} & \frac{-1+T_1}{-T_1-T_2+T_1 T_2} & 0 \\
t_2 & -\frac{T_1(-1+T_2) T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & -\frac{(1-2 T_1-T_2+T_1 T_2) T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & \frac{-1+T_2}{-T_2-T_3+T_2 T_3} \\
t_3 & \frac{T_1(-1+T_2) T_2(-1+T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & \frac{T_2(1-2 T_1-T_2+T_1 T_2)(-1+T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & \frac{1-2 T_2-T_3+T_2 T_3}{-T_2-T_3+T_2 T_3}
\end{pmatrix}$$

$\gamma = \mathbf{Rp}_{16}\,\mathbf{Rm}_{74}\,\mathbf{Rm}_{52}\,\mathbf{Rp}_{8,11}\,\mathbf{Rm}_{12,9}\,\mathbf{Rm}_{10,3};$

$\gamma \,//\, \mathbf{m}_{14\to1} \,//\, \mathbf{m}_{15\to1} \,//\, \mathbf{m}_{16\to1} \,//\, \mathbf{m}_{27\to2} \,//\, \mathbf{m}_{28\to2} \,//\, \mathbf{m}_{29\to2} \,//\, \mathbf{m}_{2,10\to2} \,//\, \mathbf{m}_{2,11\to2} \,//\, \mathbf{m}_{3,12\to3}$

$$\begin{pmatrix}
\frac{(-1+T_1+T_2)(-1+T_2+T_3)}{T_2 T_3} & h_1 & h_2 & h_3 \\
t_1 & \frac{T_1}{-1+T_1+T_2} & \frac{(-1+T_1) T_2}{-1+T_1+T_2} & 0 \\
t_2 & \frac{(-1+T_2) T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} & -\frac{T_2(1-T_1-2 T_2+T_1 T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2) T_3}{-1+T_2+T_3} \\
t_3 & \frac{(-1+T_2)(-1+T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & -\frac{(1-T_1-2 T_2+T_1 T_2)(-1+T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & -\frac{1-T_2-2 T_3+T_2 T_3}{-1+T_2+T_3}
\end{pmatrix}$$

$\gamma = \mathbf{Rm}_{16}\,\mathbf{Rp}_{74}\,\mathbf{Rp}_{52}\,\mathbf{Rm}_{8,11}\,\mathbf{Rp}_{12,9}\,\mathbf{Rp}_{10,3};$

$\gamma \,//\, \mathbf{m}_{41\to1} \,//\, \mathbf{m}_{51\to1} \,//\, \mathbf{m}_{61\to1} \,//\, \mathbf{m}_{72\to2} \,//\, \mathbf{m}_{82\to2} \,//\, \mathbf{m}_{92\to2} \,//\, \mathbf{m}_{10,2\to2} \,//\, \mathbf{m}_{11,2\to2} \,//\, \mathbf{m}_{12,3\to3}$

$$\begin{pmatrix}
(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3) & h_1 & h_2 & h_3 \\
t_1 & \frac{1-T_1-2 T_2+T_1 T_2}{-T_1-T_2+T_1 T_2} & \frac{(-1+T_1) T_2(1-T_2-2 T_3+T_2 T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & \frac{(-1+T_1)(-1+T_2) T_2 T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} \\
t_2 & \frac{-1+T_2}{-T_1-T_2+T_1 T_2} & -\frac{T_1(1-T_2-2 T_3+T_2 T_3)}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} & -\frac{T_1(-1+T_2) T_3}{(-T_1-T_2+T_1 T_2)(-T_2-T_3+T_2 T_3)} \\
t_3 & 0 & \frac{-1+T_3}{-T_2-T_3+T_2 T_3} & -\frac{T_2}{-T_2-T_3+T_2 T_3}
\end{pmatrix}$$

```
γ = Rp16 Rm74 Rm52 Rp8,11 Rm12,9 Rm10,3;
γ1 = γ // m41→1 // m51→1 // m61→1 // m72→2 // m82→2 // m92→2 // m10,2→2 // m11,2→2 // m12,3→3
```

$$\begin{pmatrix} \frac{(-1+T_1+T_2)(-1+T_2+T_3)}{T_1 T_2^2 T_3} & h_1 & h_2 & h_3 \\ t_1 & -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ t_2 & \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2)T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ t_3 & 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}$$

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{γ1 // m23→2, γ1 // m32→2}
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$$\left\{\begin{pmatrix} \frac{1-T_1-2T_2+2T_1T_2+T_2^2}{T_1T_2^3} & h_1 & h_2 \\ t_1 & -\frac{-1+T_1+2T_2-3T_1T_2-T_2^2+T_1T_2^2}{1-T_1-2T_2+2T_1T_2+T_2^2} & -\frac{(-1+T_1)\left(1-3T_2+T_2^2\right)}{1-T_1-2T_2+2T_1T_2+T_2^2} \\ t_2 & \frac{T_1(-1+T_2)T_2}{1-T_1-2T_2+2T_1T_2+T_2^2} & \frac{T_2(1-T_1+T_1T_2)}{1-T_1-2T_2+2T_1T_2+T_2^2} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{(-1+T_1+T_2)\left(1-3T_2+T_2^2\right)}{T_1T_2^3} & h_1 & h_2 \\ t_1 & -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & \frac{-1+T_1}{-1+T_1+T_2} \\ t_2 & \frac{T_1(-1+T_2)}{-1+T_1+T_2} & \frac{T_2}{-1+T_1+T_2} \end{pmatrix}\right\}$$

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2)T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix};$$

$$\omega = \frac{(-1+T_1+T_2)(-1+T_2+T_3)}{T_1T_2^2T_3};$$

```
-β (ϵ + θ) + γ (ϵ + θ) + ϵ ϕ - θ ψ /. T3 → T2 // Simplify
```

$$\frac{(1+T_1(-2+T_2))(-1+T_2)}{(-1+T_1+T_2)(-1+2T_2)}$$

$$\begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2)T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}.\{1, 1, 1\} \text{ // FullSimplify}$$

$$\left\{\frac{3-2T_3+T_2(-5+T_2+2T_3)-T_1(4-3T_3+T_2(-6+T_2+2T_3))}{(-1+T_1+T_2)(-1+T_2+T_3)},\right.$$
$$\left.\frac{T_1(-1+T_2)(-1+T_2+T_3)+T_2(-2-T_2(-3+T_3)+T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)}, \frac{T_2(-1+T_3)+T_3}{-1+T_2+T_3}\right\}$$

$$\{1,1,1\}.\begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & -\frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_1)(-1+T_2)}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_2(1-2T_2-T_3+T_2T_3)}{(-1+T_1+T_2)(-1+T_2+T_3)} & \frac{(-1+T_2)T_2}{(-1+T_1+T_2)(-1+T_2+T_3)} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix} \text{ // FullSimplify}$$

{1, 1, 1}

$$\text{Clear}[\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi,\mu,\omega];\ \Gamma\left[\omega,\{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right]$$

$$\begin{pmatrix}\omega & h_a & h_b & h_S\\ t_a & \alpha & \beta & \theta\\ t_b & \gamma & \delta & \epsilon\\ t_S & \phi & \psi & \Xi\end{pmatrix}$$

$$\Gamma\left[\omega,\{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right] \text{ // } m_{ab\to c}$$

$$\begin{pmatrix}-(-1+\beta)\,\omega & h_c & h_S\\ t_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta}\\ t_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta}\end{pmatrix}$$

$$\Gamma\left[\omega,\{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right] \text{ // } m_{ba\to c}$$

$$\begin{pmatrix}-(-1+\gamma)\,\omega & h_c & h_S\\ t_c & \frac{-\beta+\beta\gamma-\alpha\delta}{-1+\gamma} & \frac{-\alpha\epsilon-\theta+\gamma\theta}{-1+\gamma}\\ t_S & \frac{-\delta\phi-\psi+\gamma\psi}{-1+\gamma} & \frac{-\Xi+\gamma\Xi-\epsilon\phi}{-1+\gamma}\end{pmatrix}$$

$$(1-\gamma)\begin{pmatrix}\frac{-\beta+\beta\gamma-\alpha\delta}{-1+\gamma} & \frac{-\alpha\epsilon-\theta+\gamma\theta}{-1+\gamma}\\ \frac{-\delta\phi-\psi+\gamma\psi}{-1+\gamma} & \frac{-\Xi+\gamma\Xi-\epsilon\phi}{-1+\gamma}\end{pmatrix} \text{ // Simplify // MatrixForm}$$

$$\begin{pmatrix}\beta-\beta\gamma+\alpha\delta & \alpha\epsilon+\theta-\gamma\theta\\ \delta\phi+\psi-\gamma\psi & \Xi-\gamma\Xi+\epsilon\phi\end{pmatrix}$$

$$\text{FullSimplify}[(\Xi+\epsilon\phi/(1-\gamma)-1)(1-\gamma)-(\Xi+\theta\psi/(1-\beta)-1)(1-\beta)\ /.\ \{\Xi\to 1-\epsilon-\theta\}]$$

$$-\beta\,(\epsilon+\theta)+\gamma\,(\epsilon+\theta)+\epsilon\,\phi-\theta\,\psi$$

$$\text{FullSimplify}[(\Xi+\epsilon\phi/(1-\gamma)-1)-(\Xi+\theta\psi/(1-\beta)-1)\ /.\ \{\Xi\to 1-\epsilon-\theta\}]$$

$$\frac{\epsilon\,\phi}{1-\gamma}+\frac{\theta\,\psi}{-1+\beta}$$

```
ΓZ[L_] := Module[{s, Z, c, k},
   s = Skeleton[L];
   Z =
    Times @@ PD[L] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], Rp_{l i}, Rm_{j i}];
   Do[Z = Z // m_{s⟦c,1⟧, s⟦c,k⟧ → s⟦c,1⟧}, {c, Length[s]}, {k, 2, Length[s⟦c⟧]}];
   Z];
ΓA[K_] := ΓZ[K]⟦1⟧ /. T_ :> T;
ΓMVA[L_Link] := Module[{Hs, ω, μ, A},
  {ω, μ} = List @@ ΓZ[L];
  Hs = Rest[h_# & /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h_a_ :> t_a];
  Factor[ω Det[A - IdentityMatrix@Length@Hs] / (1 - T_{Skeleton[L]⟦1,1⟧})]
 ]
```

```
ΓZ[Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\begin{pmatrix} \frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9} & h_1 \\ t_1 & \frac{T_9 \left(1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9+\right.}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1} \\ t_5 & \frac{T_1 (-1+T_5)(1-T_1+T_1 T_5)(-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1} \\ t_9 & \frac{(-1+T_5) T_5 (-1+T_9)(1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1} \end{pmatrix}$$

```
ΓA[Knot[8, 17]]
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.

$$-\frac{1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6}{T^2}$$

```
ΓMVA[Link["L6a4"]]
```

$$-\frac{(-1+T_1)(-1+T_5)(-1+T_9)}{T_1 T_5}$$

```
Factor[Alexander[#][T] / ΓA[#]] & /@ AllKnots[{3, 9}]
```

$$\left\{T, \frac{1}{T}, T^2, T^2, 1, 1, 1, T^3, T^3, \frac{1}{T^4}, \frac{1}{T^4}, T^3, T, \frac{1}{T^2}, T, T, \frac{1}{T}, \frac{1}{T}, \frac{1}{T^3}, T, \frac{1}{T}, \frac{1}{T}, \frac{1}{T}, \frac{1}{T}, T, \frac{1}{T}, \frac{1}{T}, T, T^3, T, \frac{1}{T}, 1, \frac{1}{T^4}, 1, T, T^4, T^4, \frac{1}{T^5}, T^4, \frac{1}{T^5}, T^4, T^4, 1, T^4, \frac{1}{T^5}, \frac{1}{T^3}, T^2, \frac{1}{T^5}, \frac{1}{T^3}, \frac{1}{T^3}, \frac{1}{T^5}, \frac{1}{T}, T^4, \frac{1}{T}, T^2, \frac{1}{T^3}, \frac{1}{T^2}, T^4, 1, T^2, \frac{1}{T^3}, T, T, \frac{1}{T}, 1, T, \frac{1}{T^3}, 1, 1, T^4, \frac{1}{T^3}, \frac{1}{T}, T^4, \frac{1}{T^4}, 1, 1, \frac{1}{T^2}, \frac{1}{T^3}, 1, T^2, 1, \frac{1}{T^2}, \frac{1}{T^4}, \frac{1}{T^6}\right\}$$

```
Factor[1/ΓMVA[#] (MultivariableAlexander[#][T] /. T[i_] :> T_Skeleton[#][[i,1]])] & /@
 AllLinks[{2, 8}]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\left\{-T_1^2 T_3, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1}\, T_5^{3/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^2 T_7^2, -T_1^2 T_7^2, -\frac{\sqrt{T_1}\,\sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2}, -\frac{\sqrt{T_1}\,\sqrt{T_5}}{T_9^{3/2}}, -\sqrt{T_1}\,\sqrt{T_5}, -T_1^{3/2} T_5^{7/2}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -T_1 T_7^2, -\frac{1}{T_7}, -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{7/2}, -\sqrt{T_1}\, T_5^{5/2}, -\sqrt{T_1}\, T_5^{3/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1}\, T_5^{3/2}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{7/2}, -\frac{T_1}{T_7}, -T_1 T_7, -T_1^2 T_7^3, -T_1^2 T_7^3, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{3/2} T_5^{3/2} \sqrt{T_9}, -\sqrt{T_1}, -T_1^{3/2} T_5^2 T_{11}^2, -\frac{\sqrt{T_1}}{T_{11}^2}, -\frac{\sqrt{T_1}\, T_5}{T_{11}}, -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{3/2}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1}\,\sqrt{T_5}, -T_1^{3/2} T_5^2 T_{11}^2, -\sqrt{T_1}\, T_5^2 T_{11}, -\sqrt{T_1}\, T_5^{3/2} T_{11}^{3/2}, -T_1^{3/2} T_5^{5/2} \sqrt{T_{13}}, -\frac{\sqrt{T_1}\,\sqrt{T_5}}{\sqrt{T_9}\,\sqrt{T_{13}}}, -\sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_9}\,\sqrt{T_{13}}\right\}$$