Pensieve header: Zw and Ж // Zw for virtual knots with two crossings.

```
Options[SelectedNotebook[], InputAliases]
```

```
{InputAliases → {Zhe → Ж, zhe → ж}}
```

```
Жж // FullForm
```

```
\:0416\:0436
```

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2014-11"];
Get["../Projects/WKO4/FreeLie.m"];
Get["../Projects/WKO4/AwCalculus.m"];
$SeriesShowDegree = 2;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, ASeries, AW, b, BCH, BooleanSequence, BracketForm,
  BS, CC, Crop, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries,
  div, EulerE, Exp, InvertLieMorphism, j, J, JA, LieDerivation, LieMorphism,
  LieSeries, LS, LW, LyndonFactorization, New, RandomCWSeries, Randomizer,
  RandomLieSeries, RC, Support, tb, TopBracketForm, tr, Γ, ι, Λ, ⟝, ⟞}.
```

```
AwCalculus` implements / extends
 {*, **, E, ≡, dA, deg, dm, dS, dΔ, dη, dσ, E1, Es, hA, hm, hS, hσ, tA, tha, tm, tS, tσ, Γ, Λ}.
```

```
Z_{t:(1|s)}[K1_ ** K2_] := Z_t[K1] ** Z_t[K2];
Z_{t:(1|s)}[K1_ K2_] := Z_t[K1] Z_t[K2];
```

```
ЖZ_{t:(1|s)}[K1_ ** K2_] := ЖZ_t[K1] ** ЖZ_t[K2];
ЖZ_{t:(1|s)}[K1_ K2_] := ЖZ_t[K1] ЖZ_t[K2];
```

```
R^+[a_, b_] // Z_{t:(1|s)} := E_t[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R^-[a_, b_] // Z_{t:(1|s)} := E_t[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]]
```

```
Zw[Tangle[skel_, diag_]] := Module[{res, c, k},
  res = Z_s[diag /. {X^+ → R^+, X^- → R^-}];
  For[c = 1, c ≤ Length[skel], ++c,
   For[k = 2, k ≤ Length[skel[[c]]], ++k,
    res = res // dm[skel[[c, 1]], skel[[c, k]], skel[[c, 1]]]
   ]];
  res
 ]
```

```
Zw[Tangle[
   {Strand[r, 1, 2, 3], Strand[g, 4, 5, 6], Strand[b, 7, 8, 9]},
   X^-[r, 6] X^+[2, 4] X^-[g, 9] X^+[5, 7] X^-[b, 3] X^+[8, 1]
  ]]@{3}
```

$$E_s\Big[\Big\langle b \to LS\Big[0, \overline{gr}, \frac{1}{2}\,\overline{g\overline{gr}} + \overline{\overline{br}g} + \frac{1}{2}\,\overline{\overline{gr}r}, \ldots\Big],\ g \to LS\Big[0, -\overline{br}, \frac{1}{2}\,\overline{b\overline{br}} - \overline{b\overline{gr}} - \overline{\overline{br}g} + \frac{1}{2}\,\overline{\overline{br}r}, \ldots\Big],\ r \to LS\Big[0, \overline{bg}, \frac{1}{2}\,\overline{b\overline{bg}} + \overline{b\overline{gr}} + \frac{1}{2}\,\overline{\overline{bg}g}, \ldots\Big]\Big\rangle, CWS\Big[0, 0, 2\,\widehat{bgr}, \ldots\Big]\Big]$$

```
k = 0;
Outer[{++k , #1 * #2} → (Zw[Tangle[{Strand[1, 2, 3, 4]}, #1 * #2]]) &,
   {X⁺[1, 3], X⁻[1, 3], X⁺[3, 1], X⁻[3, 1]},
   {X⁺[2, 4], X⁻[2, 4], X⁺[4, 2], X⁻[4, 2]}
  ] // Flatten // ColumnForm
```

$$\{1, X^+[1,3]\,X^+[2,4]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS\left[2\,\widehat{1},\ 0,\ \ldots\right]\right]$$
$$\{2, X^-[2,4]\,X^+[1,3]\} \to E_s[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS[0,\ 0,\ \ldots]]$$
$$\{3, X^+[1,3]\,X^+[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS\left[\widehat{1},\ \widehat{\overline{11}},\ \ldots\right]\right]$$
$$\{4, X^-[4,2]\,X^+[1,3]\} \to E_s\left[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS\left[\widehat{1},\ -\widehat{\overline{11}},\ \ldots\right]\right]$$
$$\{5, X^-[1,3]\,X^+[2,4]\} \to E_s[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS[0,\ 0,\ \ldots]]$$
$$\{6, X^-[1,3]\,X^-[2,4]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS\left[-2\,\widehat{1},\ 0,\ \ldots\right]\right]$$
$$\{7, X^-[1,3]\,X^+[4,2]\} \to E_s\left[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS\left[-\widehat{1},\ -\widehat{\overline{11}},\ \ldots\right]\right]$$
$$\{8, X^-[1,3]\,X^-[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS\left[-\widehat{1},\ \widehat{\overline{11}},\ \ldots\right]\right]$$
$$\{9, X^+[2,4]\,X^+[3,1]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS\left[\widehat{1},\ 0,\ \ldots\right]\right]$$
$$\{10, X^-[2,4]\,X^+[3,1]\} \to E_s\left[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS\left[-\widehat{1},\ 0,\ \ldots\right]\right]$$
$$\{11, X^+[3,1]\,X^+[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS[0,\ 0,\ \ldots]\right]$$
$$\{12, X^-[4,2]\,X^+[3,1]\} \to E_s[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS[0,\ 0,\ \ldots]]$$
$$\{13, X^-[3,1]\,X^+[2,4]\} \to E_s\left[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS\left[\widehat{1},\ 0,\ \ldots\right]\right]$$
$$\{14, X^-[2,4]\,X^-[3,1]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS\left[-\widehat{1},\ 0,\ \ldots\right]\right]$$
$$\{15, X^-[3,1]\,X^+[4,2]\} \to E_s[\langle 1 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS[0,\ 0,\ \ldots]]$$
$$\{16, X^-[3,1]\,X^-[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1},\ 0,\ \ldots\right]\right\rangle,\ CWS[0,\ 0,\ \ldots]\right]$$

```
R⁺[a_, b_] // %Z_s := (R⁺[q, b] ** R⁺[a, b] ** R⁻[q, a]) // Z_s // hη[q];
R⁻[a_, b_] // %Z_s := (R⁺[q, a] ** R⁻[a, b] ** R⁻[q, b]) // Z_s // hη[q];

R⁺[1, 2] // %Z_s
```

$$E_s\left[\left\langle 1 \to LS\left[-\overline{q},\ 0,\ \ldots\right],\ 2 \to LS\left[\overline{1}+\overline{q},\ \frac{\overline{1\,q}}{2},\ \ldots\right]\right\rangle,\ CWS[0,\ 0,\ \ldots]\right]$$

```
R⁺[1, 2] ** R⁻[1, 2] // %Z_s
```

$$E_s[\langle 1 \to LS[0,\ 0,\ \ldots],\ 2 \to LS[0,\ 0,\ \ldots]\rangle,\ CWS[0,\ 0,\ \ldots]]$$

```
lhs = R⁺[1, 2] ** R⁺[1, 3] ** R⁺[2, 3] // %Z_s; rhs = R⁺[2, 3] ** R⁺[1, 3] ** R⁺[1, 2] // %Z_s;
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

$$\left\{E_s\left[\left\langle 1 \to LS\left[-2\,\overline{q},\ 0,\ 0,\ \ldots\right],\ 2 \to LS\left[\overline{1},\ \overline{1\,q},\ \frac{1}{2}\,\overline{\overline{1\,q}\,q},\ \ldots\right],\right.\right.$$
$$3 \to LS\left[\overline{1}+\overline{2}+2\,\overline{q},\ \frac{\overline{1\,2}}{2}+\overline{1\,q},\ \frac{1}{12}\,\overline{1\,\overline{1\,2}}+\frac{1}{6}\,\overline{1\,\overline{1\,q}}+\frac{1}{6}\,\overline{1\,\overline{2\,q}}+\frac{1}{6}\,\overline{2\,\overline{2\,q}}+\frac{1}{12}\,\overline{\overline{1\,2}\,2}+$$
$$\left.\left.\frac{1}{3}\,\overline{\overline{1\,q}\,2}+\frac{1}{3}\,\overline{\overline{1\,q}\,q}-\frac{1}{6}\,\overline{\overline{2\,q}\,q},\ \ldots\right]\right\rangle,\ CWS[0,\ 0,\ 0,\ \ldots]\Big],\ BS[6\ \text{True},\ \ldots]\Big\}$$

```
XZw[Tangle[skel_, diag_]] := Module[{res, c, k},
  res = XZs[diag /. {X+ → R+, X- → R-}];
  For[c = 1, c ≤ Length[skel], ++c,
   For[k = 2, k ≤ Length[skel[[c]]], ++k,
    res = res // dm[skel[[c, 1]], skel[[c, k]], skel[[c, 1]]]
   ]];
  res
 ]
```

```
k = 0;
Outer[{++k , #1 * #2} → (ZK[k] = XZw[Tangle[{Strand[1, 2, 3, 4]}, #1 * #2]]) &,
   {X+[1, 3], X-[1, 3], X+[3, 1], X-[3, 1]},
   {X+[2, 4], X-[2, 4], X+[4, 2], X-[4, 2]}
  ] // Flatten // ColumnForm
```

$$\{1, X^+[1,3]\,X^+[2,4]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS\left[2\,\widehat{1}, 0, \ldots\right]\right]$$
$$\{2, X^-[2,4]\,X^+[1,3]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS[0, 0, \ldots]]$$
$$\{3, X^+[1,3]\,X^+[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS\left[\widehat{1}, \widehat{11} + \widehat{1q}, \ldots\right]\right]$$
$$\{4, X^-[4,2]\,X^+[1,3]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS\left[\widehat{1}, -\widehat{11} - \widehat{1q}, \ldots\right]]$$
$$\{5, X^-[1,3]\,X^+[2,4]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS[0, 0, \ldots]]$$
$$\{6, X^-[1,3]\,X^-[2,4]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS\left[-2\,\widehat{1}, 0, \ldots\right]\right]$$
$$\{7, X^-[1,3]\,X^+[4,2]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS\left[-\widehat{1}, -\widehat{11} - \widehat{1q}, \ldots\right]]$$
$$\{8, X^-[1,3]\,X^-[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS\left[-\widehat{1}, \widehat{11} + \widehat{1q}, \ldots\right]\right]$$
$$\{9, X^+[2,4]\,X^+[3,1]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS\left[\widehat{1}, -\widehat{1q}, \ldots\right]\right]$$
$$\{10, X^-[2,4]\,X^+[3,1]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS\left[-\widehat{1}, \widehat{1q}, \ldots\right]]$$
$$\{11, X^+[3,1]\,X^+[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS[0, 0, \ldots]\right]$$
$$\{12, X^-[4,2]\,X^+[3,1]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS[0, 0, \ldots]]$$
$$\{13, X^-[3,1]\,X^+[2,4]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS\left[\widehat{1}, \widehat{1q}, \ldots\right]]$$
$$\{14, X^-[2,4]\,X^-[3,1]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS\left[-\widehat{1}, -\widehat{1q}, \ldots\right]\right]$$
$$\{15, X^-[3,1]\,X^+[4,2]\} \to E_s[\langle 1 \to LS[0, 0, \ldots]\rangle, CWS[0, 0, \ldots]]$$
$$\{16, X^-[3,1]\,X^-[4,2]\} \to E_s\left[\left\langle 1 \to LS\left[-2\,\overline{1}, 0, \ldots\right]\right\rangle, CWS[0, 0, \ldots]\right]$$

```
K1 = Tangle[{Strand[1, 2, 3, 4]}, X+[1, 2] X+[3, 4]];
```

```
(ZK[1] ≡ XZw[K1])@{4}
```

$$BS[3\,True, -LW[11q] + LW[1qq] == 0 \&\& CW[11q] + CW[1qq] == 0,$$
$$-LW[11q] + LW[1qq] == 0 \&\& 2\,LW[111q] - 2\,LW[11qq] == 0 \&\&$$
$$CW[11q] + CW[1qq] == 0 \&\& -CW[111q] + CW[11qq] - 2\,CW[1q1q] == 0, \ldots]$$

```
k = 0;
Gather[
  Outer[(++k → #1 * #2) &,
    {X⁺[1, 3], X⁻[1, 3], X⁺[3, 1], X⁻[3, 1]},
    {X⁺[2, 4], X⁻[2, 4], X⁺[4, 2], X⁻[4, 2]}
   ] // Flatten ,
  TrueQ[(%Zw[Tangle[{Strand[1, 2, 3, 4]}, #1[[2]]] ≡
       %Zw[Tangle[{Strand[1, 2, 3, 4]}, #2[[2]]])@2] &
 ] //
 ColumnForm
```

```
{1 → X⁺[1, 3] X⁺[2, 4]}
{2 → X⁻[2, 4] X⁺[1, 3], 5 → X⁻[1, 3] X⁺[2, 4], 12 → X⁻[4, 2] X⁺[3, 1], 15 → X⁻[3, 1] X⁺[4, 2]}
{3 → X⁺[1, 3] X⁺[4, 2]}
{4 → X⁻[4, 2] X⁺[1, 3]}
{6 → X⁻[1, 3] X⁻[2, 4]}
{7 → X⁻[1, 3] X⁺[4, 2]}
{8 → X⁻[1, 3] X⁻[4, 2]}
{9 → X⁺[2, 4] X⁺[3, 1]}
{10 → X⁻[2, 4] X⁺[3, 1]}
{11 → X⁺[3, 1] X⁺[4, 2]}
{13 → X⁻[3, 1] X⁺[2, 4]}
{14 → X⁻[2, 4] X⁻[3, 1]}
{16 → X⁻[3, 1] X⁻[4, 2]}
```

```
Print /@ Gather[
    Flatten[{
      Outer[(#1 * #2) &,
        {X⁺[1, 3], X⁻[1, 3], X⁺[3, 1], X⁻[3, 1]},
        {X⁺[2, 4], X⁻[2, 4], X⁺[4, 2], X⁻[4, 2]}],
      Outer[(#1 * #2) &,
        {X⁺[1, 2], X⁻[1, 2], X⁺[2, 1], X⁻[2, 1]},
        {X⁺[3, 4], X⁻[3, 4], X⁺[4, 3], X⁻[4, 3]}],
      Outer[(#1 * #2) &,
        {X⁺[1, 4], X⁻[1, 4], X⁺[4, 1], X⁻[4, 1]},
        {X⁺[2, 3], X⁻[2, 3], X⁺[3, 2], X⁻[3, 2]}]
     }],
    TrueQ[(%Zw[Tangle[{Strand[1, 2, 3, 4]}, #1]] ≡
         %Zw[Tangle[{Strand[1, 2, 3, 4]}, #2]])@4] &
  ];
```

```
{X⁺[1, 3] X⁺[2, 4]}

{X⁻[2, 4] X⁺[1, 3], X⁻[1, 3] X⁺[2, 4], X⁻[4, 2] X⁺[3, 1], X⁻[3, 1] X⁺[4, 2],
 X⁻[3, 4] X⁺[1, 2], X⁻[1, 2] X⁺[3, 4], X⁻[4, 3] X⁺[2, 1], X⁻[2, 1] X⁺[4, 3],
 X⁻[2, 3] X⁺[1, 4], X⁻[1, 4] X⁺[2, 3], X⁻[3, 2] X⁺[4, 1], X⁻[4, 1] X⁺[3, 2]}

{X⁺[1, 3] X⁺[4, 2]}

{X⁻[4, 2] X⁺[1, 3]}

{X⁻[1, 3] X⁻[2, 4]}

{X⁻[1, 3] X⁺[4, 2]}

{X⁻[1, 3] X⁻[4, 2]}

{X⁺[2, 4] X⁺[3, 1]}

{X⁻[2, 4] X⁺[3, 1]}

{X⁺[3, 1] X⁺[4, 2]}

{X⁻[3, 1] X⁺[2, 4]}

{X⁻[2, 4] X⁻[3, 1]}

{X⁻[3, 1] X⁻[4, 2]}

{X⁺[1, 2] X⁺[3, 4], X⁺[1, 4] X⁺[2, 3]}

{X⁺[1, 2] X⁺[4, 3], X⁺[2, 1] X⁺[3, 4], X⁺[1, 4] X⁺[3, 2], X⁺[2, 3] X⁺[4, 1]}

{X⁻[4, 3] X⁺[1, 2], X⁻[2, 1] X⁺[3, 4], X⁻[3, 2] X⁺[1, 4], X⁻[4, 1] X⁺[2, 3]}

{X⁻[1, 2] X⁻[3, 4], X⁻[1, 4] X⁻[2, 3]}

{X⁻[1, 2] X⁺[4, 3], X⁻[3, 4] X⁺[2, 1], X⁻[1, 4] X⁺[3, 2], X⁻[2, 3] X⁺[4, 1]}

{X⁻[1, 2] X⁻[4, 3], X⁻[2, 1] X⁻[3, 4], X⁻[1, 4] X⁻[3, 2], X⁻[2, 3] X⁻[4, 1]}

{X⁺[2, 1] X⁺[4, 3], X⁺[3, 2] X⁺[4, 1]}

{X⁻[2, 1] X⁻[4, 3], X⁻[3, 2] X⁻[4, 1]}
```

```
Print /@ Gather[
   Flatten[{
     Outer[(#1 * #2) &,
      {X⁺[1, 3], X⁻[1, 3], X⁺[3, 1], X⁻[3, 1]},
      {X⁺[2, 4], X⁻[2, 4], X⁺[4, 2], X⁻[4, 2]}],
     Outer[(#1 * #2) &,
      {X⁺[1, 2], X⁻[1, 2], X⁺[2, 1], X⁻[2, 1]},
      {X⁺[3, 4], X⁻[3, 4], X⁺[4, 3], X⁻[4, 3]}],
     Outer[(#1 * #2) &,
      {X⁺[1, 4], X⁻[1, 4], X⁺[4, 1], X⁻[4, 1]},
      {X⁺[2, 3], X⁻[2, 3], X⁺[3, 2], X⁻[3, 2]}]
    }],
   TrueQ[(Zw[Tangle[{Strand[1, 2, 3, 4]}, #1]] ≡
        Zw[Tangle[{Strand[1, 2, 3, 4]}, #2]])@4] &
  ];
```

```
{X⁺[1, 3] X⁺[2, 4], X⁺[1, 2] X⁺[3, 4], X⁺[1, 4] X⁺[2, 3]}

{X⁻[2, 4] X⁺[1, 3], X⁻[1, 3] X⁺[2, 4], X⁻[4, 2] X⁺[3, 1], X⁻[3, 1] X⁺[4, 2],
 X⁻[3, 4] X⁺[1, 2], X⁻[1, 2] X⁺[3, 4], X⁻[4, 3] X⁺[2, 1], X⁻[2, 1] X⁺[4, 3],
 X⁻[2, 3] X⁺[1, 4], X⁻[1, 4] X⁺[2, 3], X⁻[3, 2] X⁺[4, 1], X⁻[4, 1] X⁺[3, 2]}

{X⁺[1, 3] X⁺[4, 2]}

{X⁻[4, 2] X⁺[1, 3]}

{X⁻[1, 3] X⁻[2, 4], X⁻[1, 2] X⁻[3, 4], X⁻[1, 4] X⁻[2, 3]}

{X⁻[1, 3] X⁺[4, 2]}

{X⁻[1, 3] X⁻[4, 2]}

{X⁺[2, 4] X⁺[3, 1], X⁺[1, 2] X⁺[4, 3], X⁺[2, 1] X⁺[3, 4], X⁺[1, 4] X⁺[3, 2], X⁺[2, 3] X⁺[4, 1]}

{X⁻[2, 4] X⁺[3, 1], X⁻[1, 2] X⁺[4, 3], X⁻[3, 4] X⁺[2, 1], X⁻[1, 4] X⁺[3, 2], X⁻[2, 3] X⁺[4, 1]}

{X⁺[3, 1] X⁺[4, 2], X⁺[2, 1] X⁺[4, 3], X⁺[3, 2] X⁺[4, 1]}

{X⁻[3, 1] X⁺[2, 4], X⁻[4, 3] X⁺[1, 2], X⁻[2, 1] X⁺[3, 4], X⁻[3, 2] X⁺[1, 4], X⁻[4, 1] X⁺[2, 3]}

{X⁻[2, 4] X⁻[3, 1], X⁻[1, 2] X⁻[4, 3], X⁻[2, 1] X⁻[3, 4], X⁻[1, 4] X⁻[3, 2], X⁻[2, 3] X⁻[4, 1]}

{X⁻[3, 1] X⁻[4, 2], X⁻[2, 1] X⁻[4, 3], X⁻[3, 2] X⁻[4, 1]}
```