

Pensieve header: Testing the common program for all w-meta-calculi. Continues pensieve://2014-06/MetaCalculi/, continued pensieve/Projects/MetaCalculi/.

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/2014-07/MetaCalculi/"];
```

```
<< KnotTheory`
```

```
<< MetaCalculi-Program.m
```

Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.

Read more at <http://katlas.org/wiki/KnotTheory>.

General

```
SXForm[L = Link["L6a4"]]
```

KnotTheory:loading: Loading precomputed data in PD4Links`.

```
SXForm[{Loop[1, 2, 3, 4], Loop[5, 6, 7, 8], Loop[9, 10, 11, 12]},  
Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]
```

```
Z[L]
```

```
dm[9, 12, 9][
```

```
dm[9, 11, 9][dm[9, 10, 9][dm[5, 8, 5][dm[5, 7, 5][dm[5, 6, 5][dm[1, 4, 1][dm[1, 3, 1][  
dm[1, 2, 1][Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]]]]]]]]]]
```

α -Calculus

```
{Xpab, Xmab} // A
```

$$\left\{ \begin{pmatrix} 1 & h[b] \\ t[a] & \frac{-1+e^{c_a}}{c_a} \end{pmatrix}, \begin{pmatrix} 1 & h[b] \\ t[a] & \frac{e^{-c_a}(1-e^{c_a})}{c_a} \end{pmatrix} \right\}$$

```
{Xm51 Xm62 Xp34 // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
```

```
Xp61 Xm24 Xm35 // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

$$\left\{ \begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix} \right\}$$

$$\alpha = \mathbf{Xm}_{12,1} \mathbf{Xm}_{27} \mathbf{Xm}_{83} \mathbf{Xm}_{4,11} \mathbf{Xp}_{16,5} \mathbf{Xp}_{6,13} \mathbf{Xp}_{14,9} \mathbf{Xp}_{10,15} // \mathbf{A}$$

$$\begin{pmatrix} 1 & h[1] & h[3] & h[5] & h[7] & h[9] & h[11] & h[13] & h[15] \\ t[2] & 0 & 0 & 0 & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 & 0 & 0 & 0 \\ t[4] & 0 & 0 & 0 & 0 & 0 & \frac{e^{-c_4}(1-e^{c_4})}{c_4} & 0 & 0 \\ t[6] & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_6}}{c_6} & 0 \\ t[8] & 0 & \frac{e^{-c_8}(1-e^{c_8})}{c_8} & 0 & 0 & 0 & 0 & 0 & 0 \\ t[10] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{10}}}{c_{10}} \\ t[12] & \frac{e^{-c_{12}}(1-e^{c_{12}})}{c_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t[14] & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{14}}}{c_{14}} & 0 & 0 & 0 \\ t[16] & 0 & 0 & \frac{-1+e^{c_{16}}}{c_{16}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha = \mathbf{Xm}_{12,1} \mathbf{Xm}_{27} \mathbf{Xm}_{83} \mathbf{Xm}_{4,11} \mathbf{Xp}_{16,5} \mathbf{Xp}_{6,13} \mathbf{Xp}_{14,9} \mathbf{Xp}_{10,15} // \mathbf{A};$$

$$\text{Do}[\alpha = \alpha // \text{dm}[1, \mathbf{k}, 1], \{\mathbf{k}, 2, 16\}]; \alpha$$

$$\left(\begin{array}{c} e^{-3 c_1} (-1 + 4 e^{c_1} - 8 e^{2 c_1} + 11 e^{3 c_1} - 8 e^{4 c_1} + 4 e^{5 c_1} - e^{6 c_1}) \\ t[1] \end{array} \right)$$

Testing R3

$$\{ (\mathbf{Xp}_{12} // \mathbf{A}) ** (\mathbf{Xp}_{13} // \mathbf{A}) ** (\mathbf{Xp}_{23} // \mathbf{A}), (\mathbf{Xp}_{23} // \mathbf{A}) ** (\mathbf{Xp}_{13} // \mathbf{A}) ** (\mathbf{Xp}_{12} // \mathbf{A}) \}$$

$$\left\{ \begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix}, \begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix} \right\}$$

Testing the KV Solution

The Hard R4 Equation

$$\text{Print} /@ \{ (\mathbf{Xp}_{23} // \mathbf{A}) ** (\mathbf{Xp}_{13} // \mathbf{A}) ** (\mathbf{V} // \mathbf{A}), (\mathbf{V} // \mathbf{A}) ** ((\mathbf{Xp}_{13} // \mathbf{A}) // \text{d}\Delta[1, 1, 2]) \};$$

$$\begin{pmatrix}
 2^{1/4} \left(\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1} \right)^{1/4} \left(\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2} \right)^{1/4} \\
 \left(\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2} \right)^{1/4} \\
 h[1] \\
 t[1] \\
 t[2]
 \end{pmatrix}
 \begin{pmatrix}
 -\sqrt{2} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} c_2 + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} \\
 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2^2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} \\
 \sqrt{2} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} - 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} \\
 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}
 \end{pmatrix}
 \begin{pmatrix}
 -e^{\frac{c_1}{2}} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}}
 \end{pmatrix}$$

$$\begin{pmatrix}
 2^{1/4} \left(\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1} \right)^{1/4} \left(\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2} \right)^{1/4} \\
 \left(\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2} \right)^{1/4} \\
 h[1] \\
 t[1] \\
 t[2]
 \end{pmatrix}
 \begin{pmatrix}
 -\sqrt{2} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} c_2 + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} \\
 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2^2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} \\
 \sqrt{2} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} - 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} \\
 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}
 \end{pmatrix}
 \begin{pmatrix}
 -e^{\frac{c_1}{2}} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}}
 \end{pmatrix}$$

```
(Xp23 // A) ** (Xp13 // A) ** (V // A) == (V // A) ** (Xp13 // A) // dΔ[1, 1, 2] // Simplify
True
```

Unitarity

```
(V // A) ** (Vi // A) // αCollect[Simplify]
(1)
```

qΔ (“renormalized cabling”)

```
qΔ0[z_, x_, y_][α_A] := Module[{b, a},
  Times[
    V // A // dσ[1 → b[x], 2 → b[y]],
    α // dΔ[z, x, y],
    V // A // dA[1] // dA[2] // dσ[1 → a[x], 2 → a[y]]
  ] // dm[b[x], x, x] // dm[b[y], y, y] // dm[x, a[x], x] // dm[y, a[y], y]
]
```

```
Xp13 // A // qΔ0[1, 1, 2] // αCollect[Simplify]
```

$$\begin{pmatrix}
 1 & h[3] \\
 t[1] & \frac{-e^{c_2} + e^{c_1+c_2}}{c_1} \\
 t[2] & \frac{-1 + e^{c_2}}{c_2}
 \end{pmatrix}$$

`(Xp23 // A) ** (Xp13 // A)`

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2} + e^{c_1 + c_2}}{c_1} \\ t[2] & \frac{-1 + e^{c_2}}{c_2} \end{pmatrix}$$

`Xp13 // A // qDelta[1, 1, 2] // alphaCollect[Simplify]`

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2} + e^{c_1 + c_2}}{c_1 + c_2} \\ t[2] & \frac{-1 + e^{c_2}}{c_2} \end{pmatrix}$$

`Clear[w, alpha, theta, phi, E];`

`alpha0 = A[w, {t[a], t[S]}.{alpha theta}.{h[a], h[S]}];`

`alpha1 = alpha0 // qDelta[a, x, y] // alphaCollect[Simplify]`

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[S] & E & \phi & \zeta \\ t[x] & \frac{-e^{c_y} \theta c_x + e^{c_x + c_y} \theta c_x - e^{c_y} \theta c_y + e^{c_x + c_y} \theta c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x - e^{c_y} \alpha c_y + e^{c_x + c_y} \alpha c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x}{-c_x + e^{c_x + c_y} c_x} \\ t[y] & \frac{-\theta c_x + e^{c_y} \theta c_x - \theta c_y + e^{c_y} \theta c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x - \alpha c_y + e^{c_y} \alpha c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x}{-c_y + e^{c_x + c_y} c_y} \end{pmatrix}$$

`alphaCollect[(Simplify[#] /. {c_x + c_y -> c_a} &][alpha1]`

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[a] & 0 & 0 & 0 \\ t[S] & E & \phi & \phi \\ t[x] & \frac{-e^{c_y} \theta c_a + e^{c_x + c_y} \theta c_a}{-c_x + e^{c_a} c_x} & \frac{-e^{c_y} \alpha c_a + e^{c_x + c_y} \alpha c_a}{-c_x + e^{c_a} c_x} & \frac{-e^{c_y} \alpha c_a + e^{c_x + c_y} \alpha c_a}{-c_x + e^{c_a} c_x} \\ t[y] & \frac{-\theta c_a + e^{c_y} \theta c_a}{-c_y + e^{c_a} c_y} & \frac{-\alpha c_a + e^{c_y} \alpha c_a}{-c_y + e^{c_a} c_y} & \frac{-\alpha c_a + e^{c_y} \alpha c_a}{-c_y + e^{c_a} c_y} \end{pmatrix}$$

`Clear[w, alpha, theta, phi, E];`

`alpha0 = A[w, {t[a], t[S]}.{alpha theta}.{h[a], h[S]}];`

`alpha0 // qDelta[a, x, y] // alphaCollect[Simplify]`

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[S] & E & \phi & \phi \\ t[x] & \frac{-e^{c_y} \theta c_x + e^{c_x + c_y} \theta c_x - e^{c_y} \theta c_y + e^{c_x + c_y} \theta c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x - e^{c_y} \alpha c_y + e^{c_x + c_y} \alpha c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x - e^{c_y} \alpha c_y + e^{c_x + c_y} \alpha c_y}{-c_x + e^{c_x + c_y} c_x} \\ t[y] & \frac{-\theta c_x + e^{c_y} \theta c_x - \theta c_y + e^{c_y} \theta c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x - \alpha c_y + e^{c_y} \alpha c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x - \alpha c_y + e^{c_y} \alpha c_y}{-c_y + e^{c_x + c_y} c_y} \end{pmatrix}$$

Gamma-Calculus

`{Xpab, Xmap} // Gamma`

$$\left\{ \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & 1 - T_a \\ s_b & 0 & T_a \\ \Sigma & 1 & T_a \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & \frac{-1 + T_a}{T_a} \\ s_b & 0 & \frac{1}{T_a} \\ \Sigma & 1 & \frac{1}{T_a} \end{pmatrix} \right\}$$

Meta-Associativity

$$n = 4; \gamma_0 = \Gamma \left[\omega, \sum_{a=0}^n h_a \sigma_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10\ a+b} \right]$$

$$\left(\begin{array}{cccccc} \omega & s_1 & s_2 & s_3 & s_4 & \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \end{array} \right)$$

γ_0 // $dm[1, 2, 1]$ // $dm[1, 3, 1]$

$$\left(\begin{array}{cccc} \omega & (1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}) & & \\ & s_1 & & \\ & s_4 & & \\ & \Sigma & & \end{array} \right) \begin{array}{l} S_1 \\ \frac{\alpha_{31} - \alpha_{12} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{11} \alpha_{32} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_3}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \frac{\alpha_{41} - \alpha_{12} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} - \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{11} \alpha_{42} + \alpha_{13} \alpha_{21} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{42} + \alpha_{21} \alpha_{43} - \alpha_{12} \alpha_{21} \alpha_4}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \sigma_1 \sigma_2 \sigma_3 \end{array}$$

γ_0 // $dm[2, 3, 2]$ // $dm[1, 2, 1]$

$$\left(\begin{array}{cccc} \omega & (1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}) & & \\ & s_1 & & \\ & s_4 & & \\ & \Sigma & & \end{array} \right) \begin{array}{l} S_1 \\ \frac{\alpha_{31} - \alpha_{12} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{11} \alpha_{32} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_3}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \frac{\alpha_{41} - \alpha_{12} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} - \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{11} \alpha_{42} + \alpha_{13} \alpha_{21} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{42} + \alpha_{21} \alpha_{43} - \alpha_{12} \alpha_{21} \alpha_4}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \sigma_1 \sigma_2 \sigma_3 \end{array}$$

$$(\gamma_0 // dm[1, 2, 1] // dm[1, 3, 1]) == (\gamma_0 // dm[2, 3, 2] // dm[1, 2, 1])$$

True

The Mirror Properties

$$n = 3; \gamma_0 = \Gamma \left[\omega, \sum_{a=0}^n h_a \sigma_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10\ a+b} \right]$$

$$\left(\begin{array}{cccc} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{array} \right)$$

γ_0 // **Mirror** // $dm[1, 2, 1]$

$$\left(\begin{array}{cccc} \omega & \frac{\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{23} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{21} \alpha_{33} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}}{\sigma_1 \sigma_2 \sigma_3} & & \\ & s_1 & & \\ & s_3 & & \\ & \Gamma & & \end{array} \right) \begin{array}{l} S_1 \\ \frac{\alpha_{13} \alpha_{32} + \alpha_{33} - \alpha_{12} \alpha_{33} - \alpha_{11} \alpha_{32}}{-\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_3} \\ \frac{\alpha_{13} \alpha_{22} + \alpha_{23} - \alpha_{12} \alpha_{23} - \alpha_{11} \alpha_{22}}{\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{23} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{21} \alpha_{33} + \alpha_{12} \alpha_{21} \alpha_{33}} \\ \frac{1}{\sigma_1 \sigma_2} \end{array}$$

γ_0 // dm[1, 2, 1] // Mirror

$$\left(\begin{array}{c} \frac{\omega (\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{23} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{21} \alpha_{33} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33})}{\sigma_1 \sigma_2 \sigma_3} \\ S_1 \\ S_3 \\ \Gamma \end{array} \right) \begin{array}{c} S_1 \\ \frac{\alpha_{13} \alpha_{32} + \alpha_{33} - \alpha_{11}}{-\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11}} \\ \frac{\alpha_{13} \alpha_{22} + \alpha_{23} - \alpha_{11}}{\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{23} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11}} \\ \frac{1}{\sigma_1 \sigma_2} \end{array}$$

$(\gamma_0$ // Mirror // dm[1, 2, 1]) == $(\gamma_0$ // dm[1, 2, 1] // Mirror)

True

$\{n = 2; \gamma_0 = \Gamma[\omega, \sum_{a=0}^n h_a \sigma_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10 a+b}],$

$t_1 = \gamma_0$ // Mirror // dS[1], $t_2 = \gamma_0$ // dS[1] // Mirror, $t_1 == t_2$

$$\left\{ \begin{pmatrix} \omega & S_1 & S_2 \\ S_1 & \alpha_{11} & \alpha_{12} \\ S_2 & \alpha_{21} & \alpha_{22} \\ \Gamma & \sigma_1 & \sigma_2 \end{pmatrix}, \begin{pmatrix} \frac{\omega \alpha_{22}}{\sigma_2} & S_1 & S_2 \\ S_1 & \frac{-\alpha_{12} \alpha_{21} + \alpha_{11} \alpha_{22}}{\alpha_{22}} & -\frac{\alpha_{21}}{\alpha_{22}} \\ S_2 & \frac{\alpha_{12}}{\alpha_{22}} & \frac{1}{\alpha_{22}} \\ \Gamma & \sigma_1 & \frac{1}{\sigma_2} \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} \frac{\omega \alpha_{22}}{\sigma_2} & S_1 & S_2 \\ S_1 & \frac{-\alpha_{12} \alpha_{21} + \alpha_{11} \alpha_{22}}{\alpha_{22}} & \frac{\alpha_{21}}{\alpha_{22}} \\ S_2 & -\frac{\alpha_{12}}{\alpha_{22}} & \frac{1}{\alpha_{22}} \\ \Gamma & \sigma_1 & \frac{1}{\sigma_2} \end{pmatrix}, \left\{ -\frac{\alpha_{21}}{\alpha_{22}} == \frac{\alpha_{21}}{\alpha_{22}} \ \&\& \ \frac{\alpha_{12}}{\alpha_{22}} == -\frac{\alpha_{12}}{\alpha_{22}} \right\} \right\}$$

Column Sums

Clear[$\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega$];

$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}];$

γ_0 // dm[a, b, c]

$$\left(\begin{array}{c} -(-1 + \beta) \omega & S_c & S_s \\ S_c & \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ S_s & \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \\ \Sigma & \sigma_a \sigma_b & \sigma_s \end{array} \right)$$

$\{1, 1\} \cdot \left(\begin{array}{c} \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \end{array} \right)$ // Simplify

$\left\{ \frac{(-1 + \beta) \gamma + (-1 + \beta) \phi - \alpha (\delta + \psi)}{-1 + \beta}, \frac{\epsilon - \beta \epsilon + \delta \theta + \Xi - \beta \Xi + \theta \psi}{1 - \beta} \right\}$

$\{1, 1\} \cdot \left(\begin{array}{c} \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \end{array} \right) / . \{ \alpha \rightarrow s1 - \gamma - \phi, \delta \rightarrow s2 - \beta - \psi, \Xi \rightarrow s3 - \theta - \epsilon \}$ // Simplify

$\left\{ \frac{s1 (-s2 + \beta) + (-1 + s2) (\gamma + \phi)}{-1 + \beta}, \frac{s3 (-1 + \beta) + \theta - s2 \theta}{-1 + \beta} \right\}$

```
{1, 1} . 
$$\begin{pmatrix} \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta} \\ \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta} \end{pmatrix} /. \{\alpha \rightarrow s1 - \gamma - \phi, \delta \rightarrow s2 - \beta - \psi, \Xi \rightarrow s3 - \theta - \epsilon\} /. \\ s1 | s2 | s3 \rightarrow 1 // Simplify \\ \{1, 1\}$$

```

Tangle Concatenation; Γ -inversion

```
n = 3; { \\
\gamma1 = \Gamma[\omega1, \sum_{a=0}^n h_a \sigma1_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10 a+b}], \\
\gamma2 = \Gamma[\omega2, \sum_{a=0}^n h_a \sigma2_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \beta_{10 a+b}], \\
FullStitch[\gamma1, \gamma2], \gamma1 ** \gamma2, FullStitch[\gamma1, \gamma2] == \gamma1 ** \gamma2} \\
\left\{ \begin{pmatrix} \omega1 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Sigma & \sigma_{11} & \sigma_{12} & \sigma_{13} \end{pmatrix}, \begin{pmatrix} \omega2 & s_1 & s_2 & s_3 \\ s_1 & \beta_{11} & \beta_{12} & \beta_{13} \\ s_2 & \beta_{21} & \beta_{22} & \beta_{23} \\ s_3 & \beta_{31} & \beta_{32} & \beta_{33} \\ \Sigma & \sigma_{21} & \sigma_{22} & \sigma_{23} \end{pmatrix}, \right. \\
\left. \begin{pmatrix} \omega1 \omega2 & & & \\ s_1 & & & \\ s_2 & & & \\ s_3 & & & \\ \Sigma & & & \end{pmatrix} \begin{pmatrix} s_1 & & & \\ \alpha_{11} \beta_{11} + \alpha_{21} \beta_{12} + \alpha_{31} \beta_{13} & \alpha_{12} \beta_{11} + \alpha_{22} \beta_{12} + \alpha_{32} \beta_{13} & \alpha_{13} \beta_{11} + \alpha_{23} \beta_{12} + \alpha_{33} \beta_{13} \\ \alpha_{11} \beta_{21} + \alpha_{21} \beta_{22} + \alpha_{31} \beta_{23} & \alpha_{12} \beta_{21} + \alpha_{22} \beta_{22} + \alpha_{32} \beta_{23} & \alpha_{13} \beta_{21} + \alpha_{23} \beta_{22} + \alpha_{33} \beta_{23} \\ \alpha_{11} \beta_{31} + \alpha_{21} \beta_{32} + \alpha_{31} \beta_{33} & \alpha_{12} \beta_{31} + \alpha_{22} \beta_{32} + \alpha_{32} \beta_{33} & \alpha_{13} \beta_{31} + \alpha_{23} \beta_{32} + \alpha_{33} \beta_{33} \\ \sigma_{11} \sigma_{21} & \sigma_{12} \sigma_{22} & \sigma_{13} \sigma_{23} \end{pmatrix}, \right. \\
\left. \begin{pmatrix} \omega1 \omega2 & & & \\ s_1 & & & \\ s_2 & & & \\ s_3 & & & \\ \Sigma & & & \end{pmatrix} \begin{pmatrix} s_2 & & & \\ \alpha_{12} \beta_{11} + \alpha_{22} \beta_{12} + \alpha_{32} \beta_{13} & \alpha_{12} \beta_{21} + \alpha_{22} \beta_{22} + \alpha_{32} \beta_{23} & \alpha_{13} \beta_{21} + \alpha_{23} \beta_{22} + \alpha_{33} \beta_{23} \\ \alpha_{12} \beta_{31} + \alpha_{22} \beta_{32} + \alpha_{32} \beta_{33} & \alpha_{12} \beta_{31} + \alpha_{22} \beta_{32} + \alpha_{32} \beta_{33} & \alpha_{13} \beta_{31} + \alpha_{23} \beta_{32} + \alpha_{33} \beta_{33} \\ \sigma_{12} \sigma_{22} & \sigma_{12} \sigma_{22} & \sigma_{13} \sigma_{23} \end{pmatrix}, True\}
```

```
\gamma1^{-1} \\
\begin{pmatrix} \frac{1}{\omega1} & & & \\ s_1 & & & \\ s_2 & & & \\ s_3 & & & \\ \Sigma & & & \end{pmatrix} \begin{pmatrix} s_1 & & & \\ \alpha_{23} \alpha_{32} - \alpha_{22} \alpha_{33} & & & \\ \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33} & & & \\ \alpha_{23} \alpha_{31} - \alpha_{21} \alpha_{33} & & & \\ \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33} & & & \\ \alpha_{22} \alpha_{31} - \alpha_{21} \alpha_{32} & & & \\ \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33} & & & \end{pmatrix} \begin{pmatrix} s_2 & & & \\ \alpha_{13} \alpha_{32} - \alpha_{12} \alpha_{33} & & & \\ \alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33} & & & \\ \alpha_{13} \alpha_{31} - \alpha_{11} \alpha_{33} & & & \\ \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33} & & & \\ \alpha_{12} \alpha_{31} - \alpha_{11} \alpha_{32} & & & \\ \alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33} & & & \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sigma_{11}} & & & \\ \frac{1}{\sigma_{12}} & & & \end{pmatrix}
```

```
\gamma1 ** \gamma1^{-1} \\
\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & 0 \\ s_2 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 1 \\ \Sigma & 1 & 1 & 1 \end{pmatrix}
```

γ_1 // ds[1] // ds[2] // ds[3]

$$\left(\begin{array}{c} -\frac{\omega_1 (\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33})}{\sigma_{11} \sigma_{12} \sigma_{13}} \\ S_1 \\ S_2 \\ S_3 \\ \Sigma \end{array} \right) \begin{array}{c} S_1 \\ \frac{\alpha_{23} \alpha_{32} - \alpha_{22} \alpha_{33}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} \\ \frac{\alpha_{23} \alpha_{31} - \alpha_{21} \alpha_{33}}{-\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33}} \\ \frac{\alpha_{22} \alpha_{31} - \alpha_{21} \alpha_{32}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} \\ \frac{1}{\sigma_{11}} \end{array}$$

$(\gamma_1 // ds[1] // ds[2] // ds[3]) = \gamma_1^{-1} // \text{Simplify}$

$$-\frac{1}{\sigma_{11} \sigma_{12} \sigma_{13}} \omega_1 (\alpha_{13} (\alpha_{22} \alpha_{31} - \alpha_{21} \alpha_{32}) + \alpha_{12} (-\alpha_{23} \alpha_{31} + \alpha_{21} \alpha_{33}) + \alpha_{11} (\alpha_{23} \alpha_{32} - \alpha_{22} \alpha_{33})) = \frac{1}{\omega_1}$$

Other

R3

{Xm51 Xm62 Xp34 // \Gamma // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
Xp61 Xm24 Xm35 // \Gamma // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}

R3

$$\left\{ \begin{array}{c} \left(\begin{array}{cccc} 1 & S_1 & S_2 & S_3 \\ S_1 & \frac{T_3}{T_2} & 0 & 0 \\ S_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ S_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{array} \right), \left(\begin{array}{cccc} 1 & S_1 & S_2 & S_3 \\ S_1 & \frac{T_3}{T_2} & 0 & 0 \\ S_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ S_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{array} \right) \end{array} \right\}$$

$\gamma = \text{Xm}_{12,1} \text{Xm}_{27} \text{Xm}_{83} \text{Xm}_{4,11} \text{Xp}_{16,5} \text{Xp}_{6,13} \text{Xp}_{14,9} \text{Xp}_{10,15} // \Gamma$

$$\left(\begin{array}{cccccccccccccccccccc} 1 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_1 & \frac{1}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_2 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_3 & 0 & 0 & \frac{1}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+T_4}{T_4} & 0 & 0 & 0 & 0 & 0 \\ S_5 & 0 & 0 & 0 & 0 & T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - T_6 & 0 & 0 & 0 \\ S_7 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_8 & 0 & 0 & \frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 - T_{10} & 0 \\ S_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_4} & 0 & 0 & 0 & 0 & 0 \\ S_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ S_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & 0 & 0 \\ S_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - T_{14} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ S_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{10} & 0 \\ S_{16} & 0 & 0 & 0 & 0 & 1 - T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \Sigma & \frac{1}{T_{12}} & 1 & \frac{1}{T_8} & 1 & T_{16} & 1 & \frac{1}{T_2} & 1 & T_{14} & 1 & \frac{1}{T_4} & 1 & T_6 & 1 & T_{10} & 1 \end{array} \right)$$

Do[$\gamma = \gamma // \text{dm}_{1k \rightarrow 1}, \{k, 2, 10\}\}; \gamma$

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2} & S_1 & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_1 \\ S_1 & \frac{T_{14} (-T_1+T_1^2+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & \frac{(-1+T_1) (1-T_1+T_1^2) T_{14} T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & 0 & -\frac{(-1+T_1) (1-T_1+T_1^2) T_{14}}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 1-T_1 & 0 \\ S_{11} & 0 & \frac{1}{T_1} & 0 & 0 & 0 & 0 & 0 \\ S_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 1 & 0 & 0 & 0 & 0 \\ S_{13} & 0 & 0 & 0 & T_1 & 0 & 0 & 0 \\ S_{14} & -\frac{(-1+T_{14}) (-T_1+T_1^2+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1) (1-T_1+T_1^2) (-1+T_{14}) T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & 0 & \frac{(-1+T_1) (1-T_1+T_1^2) (-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}} & 1 & 0 & 0 \\ S_{15} & 0 & 0 & 0 & 0 & 0 & T_1 & 0 \\ S_{16} & -\frac{T_1 (-1+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & \frac{(-1+T_1)^2 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 0 & 1 \\ \Sigma & \frac{T_{14} T_{16}}{T_1^2 T_{12}} & \frac{1}{T_1} & 1 & T_1 & 1 & T_1 & 1 \end{pmatrix}$$

8_17

$\gamma = \text{Xm}_{12,1} \text{Xm}_{27} \text{Xm}_{83} \text{Xm}_4,_{11} \text{Xp}_{16,5} \text{Xp}_{6,13} \text{Xp}_{14,9} \text{Xp}_{10,15} // \Gamma;$

Do[$\gamma = \gamma // \text{dm}[1, k, 1], \{k, 2, 16\}\}; \gamma$

8_17

$$\begin{pmatrix} -\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3} & S_1 \\ S_1 & 1 \\ \Sigma & 1 \end{pmatrix}$$

Z[$\Gamma, \text{Link}["L6a4"]]$

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\begin{pmatrix} \frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9) (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9} & S_1 \\ S_1 & \frac{T_9 (1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9) (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)} \\ S_5 & \frac{T_1 (-1+T_5) (1-T_1+T_1 T_5) (-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9) (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)} \\ S_9 & \frac{(-1+T_5) T_5 (-1+T_9) (1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9) (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)} \\ \Sigma & 1 \end{pmatrix}$$

MVA[$\Gamma, \text{Link}["L6a4"]]$

$$-\frac{(-1+T_1) (-1+T_5) (-1+T_9)}{T_1 T_5}$$

```
Factor[ $\frac{1}{\text{MVA}[\Gamma, \#]}$  (MultivariableAlexander[#][T] /. T[i_] -> TSkeleton[#][i,1])] & /@
AllLinks[{2, 8}]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\left\{ -T_1^2 T_3, -T_1^{3/2} T_3^{3/2}, -\sqrt{T_1} T_3^{3/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^2 T_7^2, -T_1 T_7^2, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_3^{3/2} T_3^{3/2}, \right.$$

$$-\frac{\sqrt{T_1} \sqrt{T_5}}{T_3^{3/2}}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^{7/2}, -\frac{\sqrt{T_1}}{T_3^{3/2}}, -\frac{\sqrt{T_1}}{T_3^{3/2}}, -T_1 T_7^2, -\frac{1}{T_7}, -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{7/2},$$

$$-\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_3^{3/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_3^{3/2}, -\sqrt{T_1} T_3^{3/2}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{7/2}, -\frac{T_1}{T_7},$$

$$-T_1 T_7, -T_1^2 T_3^3, -T_1^2 T_3^3, -T_1^{5/2} T_5^{5/2}, -T_1^{5/2} T_5^{5/2}, -T_1^{5/2} T_5^{5/2}, -T_1^{3/2} T_3^{3/2} \sqrt{T_9}, -\sqrt{T_1}, -T_1^{3/2} T_5^2 T_{11}^2,$$

$$-\frac{\sqrt{T_1}}{T_{11}^2}, -\frac{\sqrt{T_1} T_5}{T_{11}}, -\frac{T_1^{3/2} \sqrt{T_5}}{T_{13}^{3/2}}, -T_1^{3/2} T_3^{3/2} T_3^{3/2} T_{13}^{3/2}, -T_1^{3/2} T_3^{3/2}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^2 T_{11}^2,$$

$$\left. -\sqrt{T_1} T_5^2 T_{11}^2, -\sqrt{T_1} T_5^{3/2} T_{11}^{3/2}, -T_1^{3/2} T_5^{5/2} \sqrt{T_{13}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9} \sqrt{T_{13}}}, -\sqrt{T_1} \sqrt{T_5} \sqrt{T_9} \sqrt{T_{13}} \right\}$$

$\alpha \leftrightarrow \Gamma$ Conversions

```
{Xp[1, 2] //  $\Gamma$ , Xp[1, 2] //  $\mathbf{A}$  //  $\Gamma$ }
```

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix} \right\}$$

```
{Xm[1, 2] //  $\mathbf{A}$ , Xm[1, 2] //  $\Gamma$  //  $\mathbf{A}$ }
```

$$\left\{ \begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1} (1 - e^{c_1})}{c_1} \end{pmatrix}, \begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1} (1 - e^{c_1})}{c_1} \end{pmatrix} \right\}$$

```
Clear[ $\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega$ ];
```

$$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}];$$

```
{ $\gamma_0$ ,  $\gamma_0$  //  $\mathbf{A}$ ,  $\gamma_0$  //  $\mathbf{A}$  //  $\Gamma$ , ( $\gamma_0$  //  $\mathbf{A}$  //  $\Gamma$ ) /. { $\alpha \rightarrow 1 - \gamma - \phi$ ,  $\delta \rightarrow 1 - \beta - \psi$ ,  $\Xi \rightarrow 1 - \theta - \epsilon$ }}
```

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \frac{-\alpha + \sigma_a}{c_a} & -\frac{\beta}{c_a} & -\frac{\theta}{c_a} \\ t[b] & -\frac{\gamma}{c_b} & \frac{-\delta + \sigma_b}{c_b} & -\frac{\epsilon}{c_b} \\ t[S] & -\frac{\phi}{c_s} & -\frac{\psi}{c_s} & \frac{-\Xi + \sigma_s}{c_s} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & 1 - \gamma - \phi & \beta & \theta \\ s_b & \gamma & 1 - \beta - \psi & \epsilon \\ s_s & \phi & \psi & 1 - \epsilon - \theta \\ \Sigma & 1 - \alpha - \gamma - \phi + \sigma_a & 1 - \beta - \delta - \psi + \sigma_b & 1 - \epsilon - \theta - \Xi + \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & 1 - \gamma - \phi & \beta & \theta \\ s_b & \gamma & 1 - \beta - \psi & \epsilon \\ s_s & \phi & \psi & 1 - \epsilon - \theta \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix} \right\}$$

Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];

$$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}];$$

{γ0 // dm[a, b, c] // A, γ0 // A // dm[a, b, c]} /.

{α → 1 - γ - φ, δ → 1 - β - ψ, Ξ → 1 - θ - ε}

$$\left\{ \begin{array}{l} \omega - \beta \omega \\ t[c] \\ t[S] \end{array} \begin{array}{l} h[c] \\ \frac{1-\beta-\phi+\beta\phi-\psi+\gamma\psi+\phi\psi-\sigma_a\sigma_b+\beta\sigma_a\sigma_b}{-c_c+\beta c_c} \\ \frac{\phi-\beta\phi+\psi-\gamma\psi-\phi\psi}{-c_s+\beta c_s} \end{array} \begin{array}{l} h[S] \\ \frac{\epsilon-\beta\epsilon+\theta-\beta\theta-\theta\psi}{-c_c+\beta c_c} \\ \frac{1-\beta-\epsilon+\beta\epsilon-\theta+\beta\theta+\theta\psi-\sigma_s+\beta\sigma_s}{-c_s+\beta c_s} \end{array} \right\},$$

$$\left\{ \begin{array}{l} \omega - \beta \omega \\ t[c] \\ t[S] \end{array} \begin{array}{l} h[c] \\ \frac{1-\beta-\phi+\beta\phi-\psi+\gamma\psi+\phi\psi-\sigma_a\sigma_b+\beta\sigma_a\sigma_b}{-c_c+\beta c_c} \\ \frac{\phi-\beta\phi+\psi-\gamma\psi-\phi\psi}{-c_s+\beta c_s} \end{array} \begin{array}{l} h[S] \\ \frac{\epsilon-\beta\epsilon+\theta-\beta\theta-\theta\psi}{-c_c+\beta c_c} \\ \frac{1-\beta-\epsilon+\beta\epsilon-\theta+\beta\theta+\theta\psi-\sigma_s+\beta\sigma_s}{-c_s+\beta c_s} \end{array} \right\}$$

The KV solution in Γ, starting from α

V // A // αCollect[FullSimplify]

$$\left(\begin{array}{l} 2^{1/4} \left(\frac{\sinh[\frac{c_1}{2}]}{c_1} \right)^{1/4} \left(\frac{\sinh[\frac{c_2}{2}]}{c_2} \right)^{1/4} \\ \left(\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2} \right)^{1/4} \\ t[1] \\ t[2] \end{array} \begin{array}{l} h[1] \\ \frac{-\sqrt{2} \sqrt{\frac{\sinh[\frac{c_2}{2}]}{c_2}} c_2 + 2 \sqrt{\frac{\sinh[\frac{c_1}{2}]}{c_1}} c_2 \sqrt{\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}{2 \sqrt{\frac{\sinh[\frac{c_1}{2}]}{c_1}} c_1^2 \sqrt{\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh[\frac{c_1}{2}]}{c_1}} c_1 c_2 \sqrt{\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}} \\ \frac{\sqrt{2} \sqrt{\frac{\sinh[\frac{c_2}{2}]}{c_2}} - 2 \sqrt{\frac{\sinh[\frac{c_1}{2}]}{c_1}} \sqrt{\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}{2 \sqrt{\frac{\sinh[\frac{c_1}{2}]}{c_1}} c_1 \sqrt{\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh[\frac{c_1}{2}]}{c_1}} c_2 \sqrt{\frac{\sinh[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}} \end{array} \right) \frac{e^{-\frac{c_1}{2}} c_1 \sqrt{\frac{\sinh[\frac{c_2}{2}]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{\dots}}{e^{-\frac{c_1}{2}} c_1 \sqrt{\frac{\sinh[\frac{c_2}{2}]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{\dots}}$$

V // A // Γ //

ΓCollect[Assuming[T1 > 0 && T2 > 0, (# /. {Sinh[x_] => (e^x - e^-x)/2} // FullSimplify] &]

$$\left(\begin{array}{l} \left(\frac{-1+T_1}{\text{Log}[T_1]} \right)^{1/4} \left(\frac{-1+T_2}{\text{Log}[T_2]} \right)^{1/4} \\ \left(\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]} \right)^{1/4} \\ S_1 \\ S_2 \\ \Sigma \end{array} \begin{array}{l} S_1 \\ S_2 \\ 1 \\ \sqrt{T_1} \end{array} \right) \left(\begin{array}{l} \text{Log}[T_1] \left(\text{Log}[T_2] \sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} - \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} + T_1 \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \right) \\ \text{Log}[T_1 T_2] (-1+T_1) \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \\ \text{Log}[T_2] \left(-T_1 \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} + \sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \right) \\ \text{Log}[T_1 T_2] \sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \end{array} \right) \left(\begin{array}{l} S_2 \\ \text{Log}[T_1] \left(\sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} T_2 - \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \right) \\ \text{Log}[T_1 T_2] \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\dots} \\ \text{Log}[T_1] \sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} T_2 + \text{Log}[T_2] \sqrt{\dots} \\ \text{Log}[T_1 T_2] \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\dots} \end{array} \right)$$

```
ΓSimp = Assuming[T1 > 1 && T2 > 1, (# /. {Sinh[x_] => (e^x - e^-x)/2}) // FullSimplify] &;
```

```
V // A // Γ
```

$$\left(\frac{\text{Log}[T_1 T_2] (-1+T_1) (-1+T_2)}{\text{Log}[T_1] \text{Log}[T_2] (-1+T_1 T_2)} \right)^{1/4} \begin{matrix} S_1 & S_2 \\ S_1 & \frac{\text{Log}[T_1] + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] T_1 (-1+T_2)}{(-1+T_1) (-1+T_1 T_2)}}}{\text{Log}[T_1 T_2]} \frac{\text{Log}[T_1] (-1+T_2) \left(T_1 - \sqrt{\frac{\text{Log}[T_2] \text{Log}[T_1 T_2] (-1+T_1) T_1 (-1+T_2)}{\text{Log}[T_1 T_2] (-1+T_1 T_2)}} \right)}{\text{Log}[T_1 T_2] (-1+T_2)} \\ S_2 & \frac{\text{Log}[T_2] \left(-1 + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] T_1 (-1+T_2)}{\text{Log}[T_2] (-1+T_1) (-1+T_1 T_2)}} \right)}{\text{Log}[T_1 T_2]} \frac{-\text{Log}[T_2] + T_2 \left(\text{Log}[T_2] T_1 + \sqrt{\frac{\text{Log}[T_1]^3 \text{Log}[T_2] (-1+T_1) T_1 (-1+T_2)}{\text{Log}[T_1 T_2] (-1+T_1 T_2)}} \right)}{\text{Log}[T_1 T_2] (-1+T_2)} \\ \Sigma & 1 & \sqrt{T_1} \end{matrix}$$

```
Γ[V] ** Γ[Vi]
```

$$\left(\begin{matrix} 1 \\ S_1 \\ S_2 \\ \Sigma \end{matrix} \right) \frac{1}{\text{Log}[T_1 T_2] (-1 + T_1 T_2)} \begin{matrix} S_1 \\ \frac{-\text{Log}[T_1 T_2] + T_2 \left(\sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1+T_1) T_1 (-1+T_2)}{-1+T_1 T_2}} + T_1 \left(\text{Log}[T_1 T_2] + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1+T_2)}{(-1+T_1) T_1 (-1+T_1 T_2)}} \right) - T_1 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1+T_2)}{(-1+T_1) T_1 (-1+T_1 T_2)}} \right)}{\text{Log}[T_1 T_2] (-1+T_1 T_2)} \\ \frac{(-1+T_2) \left(-(-1+T_1) T_1 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1+T_2)}{(-1+T_1) T_1 (-1+T_1 T_2)}} + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1+T_1) T_1 (-1+T_2)}{-1+T_1 T_2}} \right)}{\text{Log}[T_1 T_2] (-1+T_1) (-1+T_1 T_2)} \\ 1 \end{matrix}$$

$$\frac{1}{\text{Log}[T_1 T_2] (-1 + T_1 T_2)}$$

$$\left(-\text{Log}[T_1 T_2] + T_2 \left(\sqrt{\left(\frac{1}{-1 + T_1 T_2} \text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_1) T_1 (-1 + T_2) \right)} \right) + \right.$$

$$\left. T_1 \left(\text{Log}[T_1 T_2] + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_2)}{(-1 + T_1) T_1 (-1 + T_1 T_2)}} - \right. \right.$$

$$\left. \left. T_1 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_2)}{(-1 + T_1) T_1 (-1 + T_1 T_2)}} \right) \right) // \text{PowerExpand} // \text{Simplify}$$

1

$$\left((-1 + T_2) \left(-(-1 + T_1) T_1 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_2)}{(-1 + T_1) T_1 (-1 + T_1 T_2)}} + \sqrt{\left(\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_1) T_1 (-1 + T_2)}{-1 + T_1 T_2} \right)} \right) \right) //$$

$$\left(\text{Log}[T_1 T_2] (-1 + T_1) (-1 + T_1 T_2) \right) // \text{PowerExpand} // \text{Simplify}$$

0

dS and dA for Γ , starting from α

```
Clear[ $\alpha$ ,  $\theta$ ,  $\phi$ ,  $\Xi$ ,  $\omega$ ];
```

```
 $\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_s \sigma_s, \{t_a, t_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{h_a, h_s\}];$ 
```

```
 $((\gamma_0 // A // dS[a] // \Gamma) = (\gamma_0 // A // dA[a] // \Gamma)) /. \{\alpha \rightarrow 1 - \phi, \Xi \rightarrow 1 - \theta\}$ 
```

```
True
```

```
 $(\gamma_0 // A // dS[a] // \Gamma)$ 
```

$$\begin{pmatrix} \frac{(-1+\phi)\omega}{-1+\alpha+\phi-\sigma_a} & S_a & S_s \\ S_a & -\frac{1}{-1+\phi} & -\frac{\theta}{-1+\phi} \\ S_s & \frac{\phi}{-1+\phi} & \frac{-1+\theta+\phi}{-1+\phi} \\ \Sigma & -\frac{1}{-1+\alpha+\phi-\sigma_a} & -\frac{1-\alpha-\theta+\alpha\theta-\Xi+\alpha\Xi-\phi+\theta\phi+\Xi\phi+\sigma_a-\theta\sigma_a-\Xi\sigma_a+\sigma_s-\alpha\sigma_s-\phi\sigma_s+\sigma_a\sigma_s}{-1+\alpha+\phi-\sigma_a} \end{pmatrix}$$

```
 $(\gamma_0 // A // dA[a] // \Gamma) /. \{\phi \rightarrow 1 - \alpha, \Xi \rightarrow 1 - \theta\} // \GammaCollect$ 
```

$$\begin{pmatrix} \frac{\alpha\omega}{\sigma_a} & S_a & S_s \\ S_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ S_s & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_s \end{pmatrix}$$

```
 $(\gamma_0 // dA[a]) /. \{\phi \rightarrow 1 - \alpha, \Xi \rightarrow 1 - \theta\} // \GammaCollect$ 
```

$$\begin{pmatrix} \frac{\alpha\omega}{\sigma_a} & S_a & S_s \\ S_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ S_s & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_s \end{pmatrix}$$

```
 $(\gamma_0 // A // dA[a] // \Gamma) = (\gamma_0 // dA[a]) /. \{\phi \rightarrow 1 - \alpha, \Xi \rightarrow 1 - \theta\} // Simplify$ 
```

```
True
```

dΔ for Γ, starting from α

Clear[α, θ, φ, Ξ, ω];

γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . ($\begin{matrix} \alpha & \theta \\ \phi & \Xi \end{matrix}$) . {h_a, h_s}];

((γ0 // A // dΔ[a, b, c] // Γ) /. {α → 1 - φ, Ξ → 1 - θ}) // RCollect

$$\begin{pmatrix} \omega & S_b & S_c & S_s \\ S_b & -\frac{-\text{Log}[T_b] + \phi \text{Log}[T_b] - \text{Log}[T_c] \sigma_a}{\text{Log}[T_b] + \text{Log}[T_c]} & -\frac{\text{Log}[T_b] (-1 + \phi + \sigma_a)}{\text{Log}[T_b] + \text{Log}[T_c]} & \frac{\theta \text{Log}[T_b]}{\text{Log}[T_b] + \text{Log}[T_c]} \\ S_c & -\frac{\text{Log}[T_c] (-1 + \phi + \sigma_a)}{\text{Log}[T_b] + \text{Log}[T_c]} & \frac{\text{Log}[T_c] - \phi \text{Log}[T_c] + \text{Log}[T_b] \sigma_a}{\text{Log}[T_b] + \text{Log}[T_c]} & \frac{\theta \text{Log}[T_c]}{\text{Log}[T_b] + \text{Log}[T_c]} \\ S_s & \phi & \phi & 1 - \theta \\ \Sigma & \sigma_a & \sigma_a & \sigma_s \end{pmatrix}$$

Clear[α, θ, φ, Ξ, ω];

γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . ($\begin{matrix} \alpha & \theta \\ \phi & \Xi \end{matrix}$) . {h_a, h_s}];

γ0 // A // dΔ[a, b, c] // dS[c] // dm[b, c, a]

Power::infy: Infinite expression $\frac{1}{0}$ encountered. >>

Power::infy: Infinite expression $\frac{1}{0}$ encountered. >>

Infinity::indet: Indeterminate expression ComplexInfinity + ComplexInfinity + $\frac{(\Xi - \theta \phi - \Xi \phi - \sigma_s + \phi \sigma_s) t[S]}{-c_s + \phi c_s}$ encountered. >>

$$\left(\frac{-\omega + \phi \omega}{-1 + \alpha + \phi - \sigma_a} \right)$$

qΔ for Γ, starting from α

Clear[α, θ, φ, Ξ, ω];

γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . ($\begin{matrix} \alpha & \theta \\ \phi & \Xi \end{matrix}$) . {h_a, h_s}];

γ0 // A // qΔ[a, b, c] // Γ

$$\begin{pmatrix} \omega & S_b & S_c & \theta \\ S_b & -\frac{1 - \alpha - \phi + \alpha T_c - T_b T_c + \phi T_b T_c + \sigma_a - T_c \sigma_a}{-1 + T_b T_c} & \frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{-1 + T_b T_c} & - \\ S_c & \frac{(-1 + T_c) (\alpha - \sigma_a)}{-1 + T_b T_c} & -\frac{1 - \phi - \alpha T_c - T_b T_c + \alpha T_b T_c + \phi T_b T_c + T_c \sigma_a - T_b T_c \sigma_a}{-1 + T_b T_c} & - \\ S_s & \phi & \phi & \\ \Gamma & 1 - \alpha - \phi + \sigma_a & 1 - \alpha - \phi + \sigma_a & 1 - \theta \end{pmatrix}$$

Clear[$\alpha, \theta, \phi, \Xi, \omega$];

$\gamma_0 = \Gamma[\omega, \mathbf{h}_a \sigma_a + \mathbf{h}_s \sigma_s, \{\mathbf{t}_a, \mathbf{t}_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{\mathbf{h}_a, \mathbf{h}_s\}]$;

$\gamma_0 // \mathbf{q}\Delta[\mathbf{a}, \mathbf{b}, \mathbf{c}]$

$$\begin{pmatrix} \omega & S_b & S_c & S_s \\ S_b & \frac{-\alpha T_c + \alpha T_b T_c - \sigma_a + T_c \sigma_a}{-1 + T_b T_c} & \frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{-1 + T_b T_c} & \frac{\theta (-1 + T_b) T_c}{-1 + T_b T_c} \\ S_c & \frac{(-1 + T_c) (\alpha - \sigma_a)}{-1 + T_b T_c} & \frac{-\alpha + \alpha T_c - T_c \sigma_a + T_b T_c \sigma_a}{-1 + T_b T_c} & \frac{\theta (-1 + T_c)}{-1 + T_b T_c} \\ S_s & \phi & \phi & \Xi \\ \Gamma & \sigma_a & \sigma_a & \sigma_s \end{pmatrix}$$

Clear[$\alpha, \theta, \phi, \Xi, \omega$];

$\gamma_0 = \Gamma[\omega, \mathbf{h}_a \sigma_a + \mathbf{h}_s \sigma_s, \{\mathbf{t}_a, \mathbf{t}_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{\mathbf{h}_a, \mathbf{h}_s\}]$;

Simplify[

$(\gamma_0 // \mathbf{q}\Delta[\mathbf{a}, \mathbf{b}, \mathbf{c}])[[3]] = (\gamma_0 // \mathbf{A} // \mathbf{q}\Delta[\mathbf{a}, \mathbf{b}, \mathbf{c}] // \Gamma)[[3]] /. \{\alpha \rightarrow 1 - \phi, \theta \rightarrow 1 - \Xi\}$]

True

$\mathbf{q}\Delta$ tests for Γ

$\{\mathbf{t}_1 = \mathbf{Xp}_{13} // \Gamma // \mathbf{q}\Delta[1, 1, 2],$

$\mathbf{t}_2 = (\epsilon[1] \mathbf{Xp}_{23} // \Gamma) ** (\epsilon[2] \mathbf{Xp}_{13} // \Gamma), \mathbf{t}_1 == \mathbf{t}_2 // \text{Simplify}\}$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1 + T_1) T_2 \\ s_2 & 0 & 1 & 1 - T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Gamma & 1 & 1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1 + T_1) T_2 \\ s_2 & 0 & 1 & 1 - T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Gamma & 1 & 1 & T_1 T_2 \end{pmatrix}, \text{True} \right\}$$

$\{\mathbf{t}_1 = \mathbf{Xm}_{13} // \Gamma // \mathbf{q}\Delta[1, 1, 2], \mathbf{t}_2 = (\epsilon[2] \mathbf{Xm}_{13} // \Gamma) ** (\epsilon[1] \mathbf{Xm}_{23} // \Gamma), \mathbf{t}_1 == \mathbf{t}_2\}$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1 + T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1 + T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Gamma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1 + T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1 + T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Gamma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \text{True} \right\}$$

$\{\mathbf{t}_1 = \mathbf{Xp}_{3,1} // \Gamma // \mathbf{q}\Delta[1, 1, 2], \mathbf{t}_2 = (\epsilon[1] \mathbf{Xp}_{3,2} // \Gamma) ** (\epsilon[2] \mathbf{Xp}_{3,1} // \Gamma), \mathbf{t}_1 == \mathbf{t}_2\}$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1 - T_3 & 1 - T_3 & 1 \\ \Gamma & T_3 & T_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1 - T_3 & 1 - T_3 & 1 \\ \Gamma & T_3 & T_3 & 1 \end{pmatrix}, \text{True} \right\}$$

`{Xm[1, 2] // Γ, Xp[1, 2] // Γ // dS[1], Xp[1, 2] // Γ // dS[2]}`

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix} \right\}$$

`Xp[1, 2] // Γ // dS[1] // dS[2]`

$$\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}$$

`Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];`

$$\{\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}],$$

`t1 = γ0 // dm[a, b, c] // dS[c], t2 = γ0 // dS[a] // dS[b] // dm[b, a, c], t1 == t2}`

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{pmatrix}, \text{True} \right\}$$

`Clear[α, θ, φ, Ξ, ω];`

$$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_s \sigma_s, \{t_a, t_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{h_a, h_s\}];$$

`FullSimplify[{1, 1}.dS[a][γ0][A], And@@Thread[{1, 1}.γ0[A] == {1, 1}]]`

$$\left\{ \frac{1-\phi}{\alpha}, \frac{1-\phi}{\alpha} \right\}$$

`{1, 1}.dS[a][γ0][A] // Simplify`

$$\left\{ \frac{1-\phi}{\alpha}, \frac{\theta+\alpha\Xi-\theta\phi}{\alpha} \right\}$$

`And@@Thread[{1, 1}.γ0[A] == {1, 1}]]`

$$\alpha + \phi == 1 \ \&\& \ \theta + \Xi == 1$$

Clear[α, θ, φ, Ξ, ω];

γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . ($\begin{matrix} \alpha & \theta \\ \phi & \Xi \end{matrix}$) . {h_a, h_s}];

γ0 // dS[a] // dS[a]

$$\begin{pmatrix} \omega & s_a & s_s \\ s_a & \alpha & \theta \\ s_s & \phi & \Xi \\ \Sigma & \sigma_a & \sigma_s \end{pmatrix}$$

Clear[α, θ, φ, Ξ, ω];

γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . ($\begin{matrix} \alpha[T_a] & \theta[T_a] \\ \phi[T_a] & \Xi[T_a] \end{matrix}$) . {h_a, h_s}];

{(γ0 // dη[a]) (ε[a] // Γ), γ0 // qΔ[a, b, c] // dS[c] // dm[b, c, a],

γ0 // qΔ[a, b, c] // dS[c] // dm[c, b, a],

γ0 // qΔ[a, b, c] // dS[b] // dm[b, c, a], γ0 // qΔ[a, b, c] // dS[b] // dm[c, b, a]}

$$\left\{ \begin{pmatrix} \omega & s_a & s_s \\ s_a & 1 & 0 \\ s_s & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_s \end{pmatrix}, \begin{pmatrix} \frac{\omega \alpha[1]}{\sigma_a} & s_a & s_s \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_s & 0 & \frac{\alpha[1] \Xi[1] - \theta[1] \phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_s \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} \omega & s_a & s_s \\ s_a & 1 & \theta[1] \\ s_s & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_s \\ s_a & 1 & \theta[1] \\ s_s & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_s \end{pmatrix}, \begin{pmatrix} \frac{\omega \alpha[1]}{\sigma_a} & s_a & s_s \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_s & 0 & \frac{\alpha[1] \Xi[1] - \theta[1] \phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_s \end{pmatrix} \right\}$$

{Xp[S, a] // Γ, Xp[S, a] // Γ // qΔ[a, b, c] // dS[c] // dm[c, b, a]}

$$\left\{ \begin{pmatrix} 1 & s_a & s_s \\ s_a & T_s & 0 \\ s_s & 1 - T_s & 1 \\ \Sigma & T_s & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_s \\ s_a & 1 & 0 \\ s_s & 0 & 1 \\ \Sigma & 1 & 1 \end{pmatrix} \right\}$$

```
Clear[α, θ, φ, Ξ, ω];
γ0 = Γ[ω, ha σa + hs σs, {ta, ts} . (α θ / φ Ξ) . {ha, hs}] ;
{t1 = γ0 // qΔ[a, b, c] // dS[b] // dS[c],
 t2 = γ0 // dS[a] // qΔ[a, c, b], Simplify[t1 == t2]}
```

$$\left\{ \begin{array}{c} \frac{\alpha \omega}{\sigma_a} \\ S_b \\ S_c \\ S_s \\ \Sigma \end{array} \begin{array}{c} S_b \\ \frac{-\alpha T_b + \alpha T_b T_c - \sigma_a + T_b \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} \\ -\frac{T_b (-1 + T_c) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} \\ -\frac{\phi}{\alpha} \\ \frac{1}{\sigma_a} \end{array} \begin{array}{c} S_c \\ -\frac{(-1 + T_b) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} \\ \frac{-\alpha + \alpha T_b - T_b \sigma_a + T_b T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} \\ -\frac{\phi}{\alpha} \\ \frac{1}{\sigma_a} \end{array} \begin{array}{c} S_s \\ \frac{\theta (-1 + T_b)}{\alpha (-1 + T_b T_c)} \\ \frac{\theta T_b (-1 + T_c)}{\alpha (-1 + T_b T_c)} \\ \frac{\alpha \Xi - \theta \phi}{\alpha} \\ \sigma_s \end{array} \right\},$$

$$\left\{ \begin{array}{c} \frac{\alpha \omega}{\sigma_a} \\ S_b \\ S_c \\ S_s \\ \Sigma \end{array} \begin{array}{c} S_b \\ \frac{-\alpha T_b + \alpha T_b T_c - \sigma_a + T_b \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} \\ -\frac{T_b (-1 + T_c) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} \\ -\frac{\phi}{\alpha} \\ \frac{1}{\sigma_a} \end{array} \begin{array}{c} S_c \\ -\frac{(-1 + T_b) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} \\ \frac{-\alpha + \alpha T_b - T_b \sigma_a + T_b T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} \\ -\frac{\phi}{\alpha} \\ \frac{1}{\sigma_a} \end{array} \begin{array}{c} S_s \\ \frac{\theta (-1 + T_b)}{\alpha (-1 + T_b T_c)} \\ \frac{\theta T_b (-1 + T_c)}{\alpha (-1 + T_b T_c)} \\ \frac{\alpha \Xi - \theta \phi}{\alpha} \\ \sigma_s \end{array} \right\}, \text{True}$$

dA tests for Γ

$$\{ \mathbf{Xp}[1, 2] // \Gamma, (\mathbf{Xm}[1, 2] // \Gamma) /. T_1 \rightarrow 1/T_1, \mathbf{Xm}[1, 2] // \Gamma // dA[1] // dA[2] \}$$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix} \right\}$$

$$\{ \mathbf{Xm}[1, 2] // \Gamma, (\mathbf{Xp}[1, 2] // \Gamma) /. T_1 \rightarrow 1/T_1, \mathbf{Xp}[1, 2] // \Gamma // dA[1] // dA[2] \}$$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1 + T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1 + T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1 + T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix} \right\}$$

Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];

$$\{\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}],$$

t1 = γ0 // dm[a, b, c] // dA[c], t2 = γ0 // dA[a] // dA[b] // dm[b, a, c], t1 == t2}

$$\left(\begin{array}{cccc} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{array} \right), \left(\begin{array}{ccc} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{array} \right),$$

$$\left(\begin{array}{ccc} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{array} \right), \text{True}$$

Clear[α, θ, φ, Ξ, ω];

$$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_s \sigma_s, \{t_a, t_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{h_a, h_s\}];$$

{t1 = γ0 // qΔ[a, b, c] // dA[b] // dA[c],

t2 = γ0 // dA[a] // qΔ[a, b, c], Simplify[t1 == t2]}

$$\left(\begin{array}{ccc} \frac{\alpha\omega}{\sigma_a} & s_b & s_c & s_s \\ s_b & \frac{-\alpha+\alpha T_c-T_c\sigma_a+T_b T_c\sigma_a}{\alpha(-1+T_b T_c)\sigma_a} & -\frac{(-1+T_b)T_c(\alpha-\sigma_a)}{\alpha(-1+T_b T_c)\sigma_a} & \frac{\theta(-1+T_b)T_c}{\alpha(-1+T_b T_c)} \\ s_c & -\frac{(-1+T_c)(\alpha-\sigma_a)}{\alpha(-1+T_b T_c)\sigma_a} & \frac{-\alpha T_c+\alpha T_b T_c-\sigma_a+T_c\sigma_a}{\alpha(-1+T_b T_c)\sigma_a} & \frac{\theta(-1+T_c)}{\alpha(-1+T_b T_c)} \\ s_s & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\Xi-\theta\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_s \end{array} \right),$$

$$\left(\begin{array}{ccc} \frac{\alpha\omega}{\sigma_a} & s_b & s_c & s_s \\ s_b & \frac{-\alpha+\alpha T_c-T_c\sigma_a+T_b T_c\sigma_a}{\alpha(-1+T_b T_c)\sigma_a} & -\frac{(-1+T_b)T_c(\alpha-\sigma_a)}{\alpha(-1+T_b T_c)\sigma_a} & \frac{\theta(-1+T_b)T_c}{\alpha(-1+T_b T_c)} \\ s_c & -\frac{(-1+T_c)(\alpha-\sigma_a)}{\alpha(-1+T_b T_c)\sigma_a} & \frac{-\alpha T_c+\alpha T_b T_c-\sigma_a+T_c\sigma_a}{\alpha(-1+T_b T_c)\sigma_a} & \frac{\theta(-1+T_c)}{\alpha(-1+T_b T_c)} \\ s_s & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\Xi-\theta\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_s \end{array} \right), \text{True}$$

$$n = 4; \gamma_0 = \Gamma[\omega, \sum_{a=0}^n h_a \sigma_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10 a+b}]$$

$$\left(\begin{array}{ccccc} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{array} \right)$$

γ_0 // dA[1] // dA[2] // dA[3] // dA[4]

$$\left(\omega (\alpha_{14} \alpha_{23} \alpha_{32} \alpha_{41} - \alpha_{13} \alpha_{24} \alpha_{32} \alpha_{41} - \alpha_{14} \alpha_{22} \alpha_{33} \alpha_{41} + \alpha_{12} \alpha_{24} \alpha_{33} \alpha_{41} + \alpha_{13} \alpha_{22} \alpha_{34} \alpha_{41} - \alpha_{12} \alpha_{23} \alpha_{34} \alpha_{41} - \alpha_{14} \alpha_{23} \alpha_{31} \alpha_{42} + \alpha_{13} \alpha_{24} \alpha_{31} \alpha_{42} + \alpha_{14} \alpha_{21} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{34} \alpha_{42} + \alpha_{13} \alpha_{24} \alpha_{31} \alpha_{42} + \alpha_{14} \alpha_{21} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{24} \alpha_{33} \alpha_{42} - \alpha_{13} \alpha_{21} \alpha_{34} \alpha_{42} + \alpha_{11} \alpha_{23} \alpha_{34} \alpha_{42} + \alpha_{14} \alpha_{22} \alpha_{31} \alpha_{43} - \alpha_{12} \alpha_{24} \alpha_{31} \alpha_{43} - \alpha_{14} \alpha_{21} \alpha_{32} \alpha_{43} + \alpha_{11} \alpha_{24} \alpha_{32} \alpha_{43} + \alpha_{12} \alpha_{21} \alpha_{34} \alpha_{43} - \alpha_{11} \alpha_{22} \alpha_{34} \alpha_{43} - \alpha_{13} \alpha_{22} \alpha_{31} \alpha_{44} + \alpha_{12} \alpha_{23} \alpha_{31} \alpha_{44} + \alpha_{13} \alpha_{21} \alpha_{32} \alpha_{44} - \alpha_{11} \alpha_{23} \alpha_{32} \alpha_{44} - \alpha_{12} \alpha_{21} \alpha_{33} \alpha_{44} + \alpha_{11} \alpha_{22} \alpha_{33} \alpha_{44}) \right) == \text{Det} \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \right]$$

True

⊖ tests

$\Theta[1, 2]$ // A

$$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-c_1 + e^{\frac{s_2}{2}} c_1 - e^{\frac{s_1 + s_2}{2}} c_2 + e^{\frac{s_2}{2}} c_2}{c_1^2 + c_1 c_2} & \frac{-1 + e^{\frac{s_1 + s_2}{2}}}{c_1 + c_2} \\ t[2] & \frac{-1 + e^{\frac{s_1 + s_2}{2}}}{c_1 + c_2} & \frac{e^{\frac{s_1}{2}} c_1 - e^{\frac{s_1 + s_2}{2}} c_1 - c_2 + e^{\frac{s_1}{2}} c_2}{c_1 c_2 + c_2^2} \end{pmatrix}$$


```
(V // A) ** (Theta[1, 2] // A) ==
  (Xm[2, 1] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // FullSimplify
True

(Theta[1, 2] // A) == (Vi // A) ** (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify
Simplify::time :
  Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of
  TimeConstraint option may improve the result of simplification. >>
Simplify::time :
  Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of
  TimeConstraint option may improve the result of simplification. >>
Simplify::time :
  Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of
  TimeConstraint option may improve the result of simplification. >>
General::stop : Further output of Simplify::time will be suppressed during this calculation. >>
$Aborted

(Xp[2, 1] // A) == (V // A // dσ[1 → 2, 2 → 1]) ** (Theta[1, 2] // A) ** (Vi // A) // Simplify
True
```

Γb-Calculus

```
Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];
```

$$\{\gamma_0 = \Gamma_b[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}],$$

```
γ0 // Γ // dm[a, b, c] // Γb}
```

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} -\beta + \omega & s_c & s_s \\ s_c & \frac{-\beta \gamma + \alpha \delta + \gamma \omega}{\omega} & \frac{-\beta \epsilon + \delta \theta + \epsilon \omega}{\omega} \\ s_s & \frac{-\beta \phi + \alpha \psi + \phi \omega}{\omega} & \frac{-\beta \xi + \theta \psi + \xi \omega}{\omega} \\ \Sigma & \sigma_a \sigma_b & \sigma_s \end{pmatrix} \right\}$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &]
```

$$\left(\begin{array}{l} \frac{(\text{Log}[T_1] + \text{Log}[T_2])^{1/4} (-1 + T_1)^{1/4} (-1 + T_2)^{1/4}}{\text{Log}[T_1]^{1/4} \text{Log}[T_2]^{1/4} (-1 + T_1 T_2)^{1/4}} \\ s_1 \\ s_2 \\ \Sigma \end{array} \right) \frac{\text{Log}[T_1]^{1/4} \left(\sqrt{\text{Log}[T_1]} \sqrt{\text{Log}[T_1] + \text{Log}[T_2]} \sqrt{-1 + T_2} - \sqrt{\text{Log}[T_1]} \sqrt{\text{Log}[T_1] + \text{Log}[T_2]} T \right)}{}$$

$$\begin{aligned}
 & \mathbf{V} // \mathbf{A} // \mathbf{\Gamma b} // \mathbf{\Gamma bCollect[FullSimplify[PowerExpand[#]] \&] // \mathbf{\Gamma bCollect[} \\
 & \quad \mathbf{Assuming[c_1 > 0 \&\& c_2 > 0, FullSimplify[# /. Log[x_] \Rightarrow Log[x /. T_a \Rightarrow e^{c_a}]]] \&} \\
 & \left(\begin{array}{l}
 \frac{((c_1+c_2)(-1+T_1))^{1/4} (-1+T_2)^{1/4}}{(c_1 c_2 (-1+T_1 T_2))^{1/4}} \\
 S_1 \quad \frac{\sqrt{c_1 (c_1+c_2) (-1+T_2)} - T_1 \sqrt{c_1 (c_1+c_2) (-1+T_2)} - T_1 \sqrt{c_1 (c_1+c_2) (-1+T_2)} T_2 + T_1^2 \sqrt{c_1 (c_1+c_2) (-1+T_2)}}{(-1} \\
 S_2 \quad \frac{c_2^{1/4} (-1+T_2)^{1/4}}{\dots} \\
 \Sigma
 \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{V} // \mathbf{A} // \mathbf{\Gamma} // \mathbf{\GammaCollect[FullSimplify[PowerExpand[#]] \&] // \mathbf{\GammaCollect[} \\
 & \quad \mathbf{Assuming[c_1 > 0 \&\& c_2 > 0, FullSimplify[# /. Log[x_] \Rightarrow Log[x /. T_a \Rightarrow e^{c_a}]]] \&} \\
 & \left(\begin{array}{l}
 \frac{((c_1+c_2)(-1+T_1))^{1/4} (-1+T_2)^{1/4}}{(c_1 c_2 (-1+T_1 T_2))^{1/4}} \\
 S_1 \quad \frac{c_1 \sqrt{c_1 c_2 (-1+T_2)} + c_2 \sqrt{c_1 c_2 (-1+T_2)} + c_1 \sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}} \sqrt{-1+T_1 T_2}}{(c_1+c_2) \sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}} \sqrt{-1+T_1 T_2}} - \frac{\sqrt{c_1 c_2 (c_1+c_2) (-1+T_1)}}{(c_1+c_2) \sqrt{\dots}} \\
 S_2 \quad \frac{-c_1 \sqrt{c_1 c_2 (-1+T_2)} - c_2 \sqrt{c_1 c_2 (-1+T_2)} + c_2 \sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}} \sqrt{-1+T_1 T_2}}{(c_1+c_2) \sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}} \sqrt{-1+T_1 T_2}} - \frac{\sqrt{c_1 c_2 (c_1+c_2) (-1+T_1)}}{(c_1+c_2) \sqrt{\dots}} \\
 \Sigma \quad 1
 \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{V} // \mathbf{A} // \mathbf{\Gamma} // \mathbf{\GammaCollect[FullSimplify[PowerExpand[#]] \&] // \mathbf{\GammaCollect[} \\
 & \quad \mathbf{Assuming[d_1 > 0 \&\& d_2 > 0, FullSimplify[# /. Log[x_] \Rightarrow Log[x /. T_a \Rightarrow e^{d_a^2}]]] \&} \\
 & \left(\begin{array}{l}
 \frac{((d_1^2+d_2^2)(-1+T_1))^{1/4} (-1+T_2)^{1/4}}{(d_1^2 d_2^2 (-1+T_1 T_2))^{1/4}} \\
 S_1 \quad \frac{d_1 \left(d_1^2 d_2 \sqrt{-1+T_2} + d_2^2 \sqrt{-1+T_2} + d_1 \sqrt{\frac{-1+T_1}{T_1}} \sqrt{(d_1^2+d_2^2) (-1+T_1 T_2)} \right)}{(d_1^2+d_2^2) \sqrt{\frac{-1+T_1}{T_1}} \sqrt{(d_1^2+d_2^2) (-1+T_1 T_2)}} - \frac{d_1 \left(-d_2 \sqrt{(d_1^2+d_2^2) (-1+T_1)} \sqrt{\dots} \right)}{(d_1^2+d_2^2) \sqrt{-1+T_1 T_2}} \\
 S_2 \quad \frac{d_2 \left(d_1^2 \sqrt{-1+T_2} + d_1 d_2 \sqrt{-1+T_2} - d_2 \sqrt{\frac{-1+T_1}{T_1}} \sqrt{(d_1^2+d_2^2) (-1+T_1 T_2)} \right)}{(d_1^2+d_2^2) \sqrt{\frac{-1+T_1}{T_1}} \sqrt{(d_1^2+d_2^2) (-1+T_1 T_2)}} - \frac{d_2 \left(d_1 \sqrt{(d_1^2+d_2^2) (-1+T_1)} \sqrt{\dots} \right)}{(d_1^2+d_2^2) \sqrt{-1+T_1 T_2}} \\
 \Sigma \quad 1 \quad \sqrt{\dots}
 \end{array} \right)
 \end{aligned}$$

ΓI-Calculus

Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];

{γ0 = Γ1[ω, h_a σ_a + h_b σ_b + h_s σ_s, {t_a, t_b, t_s}]. $\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix}$. {h_a, h_b, h_s}], γ0 // dm[a, b, c]}

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \xi \\ \Gamma 1 & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} -(-1+\beta) \omega & s_c & s_s \\ s_c & \frac{-\gamma+\beta \gamma-\alpha \delta}{-1+\beta} & \frac{-\epsilon+\beta \epsilon-\delta \theta}{-1+\beta} \\ s_s & \frac{-\phi+\beta \phi-\alpha \psi}{-1+\beta} & \frac{-\xi+\beta \xi-\theta \psi}{-1+\beta} \\ \Gamma 1 & \sigma_a \sigma_b & \sigma_s \end{pmatrix} \right\}$$

Clear[α, θ, φ, Ξ, ω];

{γ0 = Γ1[ω, ha σa + hs σs, {ta, ts}. {α θ
φ Ξ}. {ha, hs}], γ0 // qΔ[a, b, c],
Γ[ω, ha σa + hs σs, {ta, ts}. {α θ
φ Ξ}. {ha, hs}] // qΔ[a, b, c]}

$$\left\{ \begin{pmatrix} \omega & S_a & S_s \\ S_a & \alpha & \theta \\ S_s & \phi & \Xi \\ \Gamma 1 & \sigma_a & \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & S_b & S_c & S_s \\ S_b & \frac{-\alpha T_c + \alpha T_b T_c - \sigma_a + T_c \sigma_a}{-1 + T_b T_c} & \frac{(-1 + T_c) T_c (\alpha - \sigma_a)}{-1 + T_b T_c} & \theta T_c \\ S_c & \frac{(-1 + T_b) (\alpha - \sigma_a)}{-1 + T_b T_c} & \frac{-\alpha + \alpha T_c - T_c \sigma_a + T_b T_c \sigma_a}{-1 + T_b T_c} & \theta \\ S_s & \frac{\phi (-1 + T_b)}{-1 + T_b T_c} & \frac{\phi (-1 + T_c)}{-1 + T_b T_c} & \Xi \\ \Gamma 1 & \sigma_a & \sigma_a & \sigma_s \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \omega & S_b & S_c & S_s \\ S_b & \frac{-\alpha T_c + \alpha T_b T_c - \sigma_a + T_c \sigma_a}{-1 + T_b T_c} & \frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{-1 + T_b T_c} & \frac{\theta (-1 + T_b) T_c}{-1 + T_b T_c} \\ S_c & \frac{(-1 + T_c) (\alpha - \sigma_a)}{-1 + T_b T_c} & \frac{-\alpha + \alpha T_c - T_c \sigma_a + T_b T_c \sigma_a}{-1 + T_b T_c} & \frac{\theta (-1 + T_c)}{-1 + T_b T_c} \\ S_s & \phi & \phi & \Xi \\ \Gamma & \sigma_a & \sigma_a & \sigma_s \end{pmatrix} \right\}$$

{Xp[1, 2] // Γ, Xp[1, 2] // Γ1}

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Gamma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_2 \\ s_2 & 0 & T_1 \\ \Gamma 1 & 1 & T_1 \end{pmatrix} \right\}$$

R3

{Xm51 Xm62 Xp34 // Γ1 // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
Xp61 Xm24 Xm35 // Γ1 // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}

R3

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1 + T_1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1 + T_1}{T_2} & \frac{-1 + T_2}{T_3} & 1 \\ \Gamma 1 & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1 + T_1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1 + T_1}{T_2} & \frac{-1 + T_2}{T_3} & 1 \\ \Gamma 1 & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix} \right\}$$

V // A // Γ1 // Γ1Collect[FullSimplify[PowerExpand[#]] &]

$$\left(\begin{array}{l} S_1 \\ S_2 \\ \Sigma \end{array} \right) \begin{array}{l} \frac{(\text{Log}[T_1] + \text{Log}[T_2])^{1/4} (-1 + T_1)^{1/4} (-1 + T_2)^{1/4}}{\text{Log}[T_1]^{1/4} \text{Log}[T_2]^{1/4} (-1 + T_1 T_2)^{1/4}} \\ \frac{\sqrt{\text{Log}[T_1]} \left(\text{Log}[T_1] \sqrt{\text{Log}[T_2]} \sqrt{-1 + T_2} + \text{Log}[T_2]^{3/2} \sqrt{-1 + T_2} + \sqrt{\text{Log}[T_1]} \sqrt{\text{Log}[T_1]} \cdot \right.}{(\text{Log}[T_1] + \text{Log}[T_2])^{3/2} \sqrt{\frac{-1 + T_1}{T_1}} \sqrt{-1 + T_1 T_2}} \\ \left. - \frac{\sqrt{\text{Log}[T_2]} \left(\sqrt{\text{Log}[T_1]} \sqrt{\text{Log}[T_1] + \text{Log}[T_2]} \sqrt{-1 + T_1} \sqrt{T_1} \sqrt{-1 + T_2} + \sqrt{\text{Log}[T_2]} \sqrt{-1 + T_2} \right)}{(\text{Log}[T_1] + \text{Log}[T_2]) (-1 + T_2) \sqrt{-1 + T_1 T_2}} \right) \\ 1 \end{array}$$

```
V // A // T1 // T1Collect[FullSimplify[PowerExpand[#]] &] // T1Collect[
Assuming[c1 > 0 && c2 > 0, FullSimplify[# /. Log[x_] => Log[x /. T_a_ => e^c_a]]] &]
```

$$\left(\begin{array}{l}
 \frac{((c_1+c_2)(-1+T_1))^{1/4} (-1+T_2)^{1/4}}{(c_1 c_2 (-1+T_1 T_2))^{1/4}} \\
 S_1 \\
 S_2 \\
 \Sigma
 \end{array} \right.
 \begin{array}{l}
 S_1 \\
 \frac{c_1 \sqrt{c_1 c_2 (-1+T_2)} + c_2 \sqrt{c_1 c_2 (-1+T_1)} + c_1 \sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}} \sqrt{-1+T_1 T_2}}{(c_1+c_2) \sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}} \sqrt{-1+T_1 T_2}} - \frac{\sqrt{c_1 (-1+T_2)} (c_2^{3/2} \sqrt{-1+T_1 T_2})}{(c_1+c_2) \sqrt{-1+T_1 T_2}} \\
 \frac{-\sqrt{c_1 c_2 (c_1+c_2) (-1+T_1)} \sqrt{T_1} \sqrt{-1+T_2} - c_2 \sqrt{-1+T_1 T_2} + c_2 T_1 \sqrt{-1+T_1 T_2}}{(c_1+c_2) (-1+T_2) \sqrt{-1+T_1 T_2}} \\
 1
 \end{array}
 \left. \right)$$