

Pensieve header: A common program for all w-meta-calculi. Continues pensieve://2014-06/MetaCalculi/, continued pensieve://Projects/MetaCalculi/.

General

```

Xpa,b := Xp[a, b]; Xma,b := Xm[a, b];

SXForm[L_] := SXForm[
  Skeleton[L],
  Times @@ PD[L] /.
  X[i_, j_, k_, l_] => If[PositiveQ[X[i, j, k, l]], Xp[l, i], Xm[j, i]]
];

Z[L_] := Z[Identity, L];
Z[χ_, L_] := Module[{s, z},
  {s, z} = List @@ SXForm[L];
  z = χ[z];
  Do[z = z // dm[s[[c, 1]], s[[c, k]], s[[c, 1]],
    {c, Length[s]}, {k, 2, Length[s[[c]]}];
  z
];

dA[a_, rest_][α_] := α // dA[a] // dA[rest];
dA[l_List] := dA @@ l;
dA[All][α_] := α // dA[dL[α]];
dS[a_, rest_][α_] := α // dS[a] // dS[rest];
dS[l_List] := dS @@ l;
dS[All][α_] := α // dS[dL[α]];

```

α -Calculus

α -calculus is really the “exact” β -calculus of pensieve://2012-05/beta5.1

Utilities

```

αSimplify[expr_] := expr // Together // ExpandDenominator // ExpandNumerator;
SetAttributes[αCollect, Listable];
αCollect[A[ω_, μ_]] := A[
  αSimplify[ω],
  Collect[μ, _h, Collect[#, _t, αSimplify] &]
];
αCollect[simp_][A[ω_, μ_]] := A[
  simp[ω],
  Collect[μ, _h, Collect[#, _t, simp] &]
];
hL[β_] := Union[Cases[β, h[s_] :=> s, ∞]];
tL[β_] := Union[Cases[β, t[s_] | c_s :=> s, ∞]];
dL[β_] := Union[hL[β], tL[β]];
αForm[A[ω_, μ_]] := Module[
  {tails, heads, mat},
  tails = tL[A[ω, μ]]; heads = hL[A[ω, μ]];
  mat = Outer[αSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
  PrependTo[mat, t /@ tails];
  mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
  MatrixForm[mat]
];
αForm[else_] := else /. β_A :=> αForm[β];
Format[A[ω_, μ_], StandardForm] := αForm[A[ω, μ]];
A /: A[ω1_, μ1_] == A[ω2_, μ2_] := Module[
  {heads, tails},
  tails = tL[{A[ω1, μ1], A[ω2, μ2]}];
  heads = hL[{A[ω1, μ1], A[ω2, μ2]}];
  (ω1 == ω2) && (
    And @@ Flatten[Outer[
      (Coefficient[μ1, t[#1] h[#2]] == Coefficient[μ2, t[#1] h[#2]]) &,
      tails, heads
    ]]
  )
]

```

The Meta-Cross-Product

The “Tails” meta-group

```

tm[x_, y_, z_][β_A] := αCollect[β /. {t[x] → t[z], t[y] → t[z], c_x → c_z, c_y → c_z}];
tΔ[z_, x_, y_][β_A] := αCollect[β /. {t[z] → t[x] + t[y], c_z → c_x + c_y}];
tη[x_][β_A] := αCollect[(β /. t[x] → 0) /. c_x → 0];
tS[x_][β_A] := αCollect[β /. {t[x] → -t[x], c_x → -c_x}];
tA[_][β_A] := αCollect[β];
tσ[rules__Rule][β_A] := αCollect[
  β /. {t[x_] ⇒ t[x /. {rules}], c_x ⇒ c_x /. {rules}}
];

```

The “Heads” meta-group

```

hm[x_, y_, z_][A[ω_, μ_]] := Module[
  {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
  A[ω, M + h[z] (γx + γy + (γx /. t[i_] ⇒ c_i) γy)] // αCollect
];
hΔ[z_, x_, y_][β_A] := αCollect[β /. h[z] → h[x] + h[y]];
hη[x_][β_A] := αCollect[β /. h[x] → 0];
hS[x_][A[ω_, μ_]] := Module[{γ},
  γ = 1 + D[μ, h[x]] /. t[s_] ⇒ c_s;
  αCollect[A[ω, μ /. h[x] → -h[x] / γ]]
];
hA[x_][β_A] := hS[x][β];
hσ[rules__Rule][β_A] := αCollect[β /. h[x_] ⇒ h[x /. {rules}]];

```

The TH \rightarrow HT and HT \rightarrow TH Swaps

```

tha[x_, y_][A[ω_, μ_]] := Module[
  {α, β, γ, δ, ε},
  α = Coefficient[μ, h[y] t[x]];
  β = D[μ, t[x]] /. h[y] → 0;
  γ = D[μ, h[y]] /. t[x] → 0;
  δ = μ /. h[y] | t[x] → 0;
  ε = 1 + c_x α;
  A[ω*ε, Plus[
    α (1 + (γ /. t[i_] → c_i) / ε) h[y] t[x],
    β (1 + (γ /. t[i_] → c_i) / ε) t[x],
    γ / ε h[y],
    δ - c_x / ε γ * β
  ]] // αCollect
];
hta[x_, y_][β_A] := β // hS[x] // tha[y, x] // hS[x];

```

The “double” meta-group

```

dm[x_, y_, z_][β_A] := β // tha[x, y] // hm[x, y, z] // tm[x, y, z];
dΔ[z_, x_, y_][β_] := β // tΔ[z, x, y] // hΔ[z, x, y];
dS[s_][β_] := β // hta[s, s] // hS[s] // tS[s];
dA[s_][β_] := β // hta[s, s] // hA[s] // tA[s];
dη[s_][β_] := β // hη[s] // tη[s];
dcap[s_][β_] := β // hta[s, s] // hη[s];
dσ[rules___][β_] := β // hσ[rules] // tσ[rules];
dσ[pl_List][β_] := Module[
  {σ, len, β1, k},
  len = Length[pl];
  β1 = β // (dσ @@ Table[i → σ[i], {i, len}]);
  Do[
    k = pl[[i, 1]];
    β1 = β1 // dσ[σ[i] → k];
    Do[
      β1 = β1 // dΔ[k, k, pl[[i, j]]],
      {j, 2, Length[pl[[i]]]}
    ],
    {i, len}
  ];
  β1
];
dσ[pl___Integer] := dσ[IntegerDigits /@ {pl}];

```

The “external” product

```
A /: A[ $\omega1\_ , \mu1\_$ ] A[ $\omega2\_ , \mu2\_$ ] := A[ $\omega1 * \omega2 , \mu1 + \mu2$ ];
```

Tangle Concatenation

```
A /:  $\alpha1\_A ** \alpha2\_A$  := Module[{S,  $\alpha$ ,  $\tau$ },
  S = dL[ $\alpha1$ ];
   $\alpha$  =  $\alpha1$  ( $\alpha2$  // do@@ ((# →  $\tau$ [#]) & /@ S));
  Do[
     $\alpha$  =  $\alpha$  // dm[s,  $\tau$ [s], s],
    {s, S}
  ];
   $\alpha$ 
]
```

The R-Matrix

```
SetAttributes[A, Listable];
A[p_Times] := A /@ p;
A[Xp[ $a\_ , b\_$ ]] :=  $\alpha$ Collect[A[1, ( $e^{c_a} - 1$ ) /  $c_a * t[a] h[b]$ ]];
A[Xm[ $a\_ , b\_$ ]] :=  $\alpha$ Collect[A[1, ( $e^{-c_a} - 1$ ) /  $c_a * t[a] h[b]$ ]];
A[ $\Theta$ [ $a\_ , b\_$ ]] := (A[1, ( $e^{c_a/2} - 1$ ) /  $c_a * t[a] h[a]$ ] // d $\Delta$ [a, a, b]) **
  (A[1, ( $e^{-c_b/2} - 1$ ) /  $c_b * t[b] h[b]$ ]);
A[ $\Theta$ i[ $a\_ , b\_$ ]] := (A[1, ( $e^{-c_a/2} - 1$ ) /  $c_a * t[a] h[a]$ ] // d $\Delta$ [a, a, b]) **
  (A[1, ( $e^{c_b/2} - 1$ ) /  $c_b * t[b] h[b]$ ])
```

The Exact KV Solution in α

Module [$\{v, \kappa, \omega, \alpha, \beta, \gamma, \delta\}$,

$$v[x_-] := \sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x/2}}; \quad \kappa[x_-] := v[x]^{-1/2}; \quad \omega = \frac{\kappa[c_1 + c_2]}{\kappa[c_1] \kappa[c_2]}$$

$$\gamma = \frac{v[c_2] - v[c_1] v[c_1 + c_2]}{(c_1 + c_2) v[c_1] v[c_1 + c_2]}; \quad \delta = \frac{e^{\frac{c_1}{2}}}{c_2} - \frac{v[c_1 + c_2] e^{c_1 + c_2} v[c_1] c_1}{(-1 + e^{c_1 + c_2}) v[c_2] c_2} - \frac{1}{c_1 + c_2};$$

$$\alpha = \frac{-c_2}{c_1} \gamma; \quad \beta = \frac{1}{c_1} \left(e^{\frac{c_1}{2}} - c_2 \delta - 1 \right);$$

A[C] = **α Collect**[**A**[κ [c₁], 0]],

A[V] = **α Collect**[**A**[ω , {t@1, t@2}. ($\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$). {h@1, h@2}]],

A[Vi] = **A**[V] // **dA**[1] // **dA**[2]

}
]

$$\left\{ \begin{array}{l} \left(\frac{1}{2^{1/4} \left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1} \right)^{1/4}} \right) \\ t[1] \\ t[2] \end{array} \right\}, \quad \left(\begin{array}{l} \frac{2^{1/4} \left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1} \right)^{1/4} \left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2} \right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2} \right)^{1/4}} \\ t[1] \\ t[2] \end{array} \right) \cdot \left(\begin{array}{l} h[1] \\ -\sqrt{2} \sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}} c_2 + 2 \sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2}} \\ 2 \sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} c_1^2 \sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2}} + 2 \sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} c_1 c_2 \sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2}} \\ \sqrt{2} \sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}} - 2 \sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} \sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2}} \\ 2 \sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2}} + 2 \sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)}\right]}{c_1+c_2}} \end{array} \right) \cdot \frac{c_1}{-e^{\frac{c_1}{2}}}$$

$q\Delta$ (“renormalized cabling”)

q Δ [a, x, y][A[ω , μ]] := **Module** [

{ $\alpha, \theta, \phi, \Xi, M$ },

$$\left(\begin{array}{l} \alpha & \theta \\ \phi & \Xi \end{array} \right) = \left(\begin{array}{ll} \partial_{h[a]} \mu & \partial_{t[a]} \mu \\ \partial_{h[a]} \mu & \mu \end{array} \right) /. (h | t)[a] \rightarrow 0;$$

$$M = \left(\begin{array}{lll} \frac{(e^{c_x} - e^{c_y}) \alpha c_x}{(-1 + e^{c_x}) c_x} & \frac{(e^{c_x} - e^{c_y}) \alpha c_x}{(-1 + e^{c_x}) c_x} & \frac{(e^{c_x} - e^{c_y}) \theta c_x}{(-1 + e^{c_x}) c_x} \\ \frac{(e^{c_x} - e^{c_y}) \alpha c_x}{(-1 + e^{c_x}) c_x} & \frac{(e^{c_x} - e^{c_y}) \alpha c_x}{(-1 + e^{c_x}) c_x} & \frac{(e^{c_x} - e^{c_y}) \theta c_x}{(-1 + e^{c_x}) c_x} \\ \phi & \phi & \Xi \end{array} \right);$$

α Collect[**A**[ω , {t[x], t[y], 1}.M.{h[x], h[y], 1}]] /. $c_a \rightarrow c_x + c_y$

];

Γ -Calculus

```

ΓSimp = Factor; SetAttributes[ΓCollect, Listable];
ΓCollect[Γ[ω_, σ_, λ_]] := ΓCollect[ΓSimp][Γ[ω, σ, λ]];
ΓCollect[simp_][Γ[ω_, σ_, λ_]] := Γ[simp[ω], simp[σ],
  Collect[λ, h_, Collect[#, t_, simp] &]];
dL[Γ[_ , _ , λ_]] := Union[Cases[λ, (h | t)_a_ => a, Infinity]];
Γ[ω1_, _ , _][ω] := ω1;
Γ[ω_, σ_, λ_][Σ] := (∂h#σ) & /@ dL[Γ[ω, σ, λ]];
Γ[ω_, σ_, λ_][A] :=
  Module[{S = dL[Γ[ω, σ, λ]], Outer[ΓSimp[(∂t#1h#2λ)] &, S, S]};
ΓForm[Γ[ω_, σ_, λ_]] := Module[{S, M},
  S = dL[Γ[ω, σ, λ]];
  M = Γ[ω, σ, λ][A] // Transpose;
  PrependTo[M, s# & /@ S];
  M = Join[
    {Prepend[s# & /@ S, ω]},
    Transpose[M],
    {Prepend[Γ[ω, σ, λ][Σ], "Γ"]}
  ];
  MatrixForm[M]
];
ΓForm[else_] := else /. Γ[ω_, σ_, λ_] => ΓForm[Γ[ω, σ, λ]];
Format[Γ[ω_, σ_, λ_], StandardForm] := ΓForm[Γ[ω, σ, λ]];
Γ /: Γ[ω1_, σ1_, μ1_] == Γ[ω2_, σ2_, μ2_] := Module[
  {S},
  S = dL[Γ[ω1, σ1, μ1]] ∪ dL[Γ[ω2, σ2, μ2]];
  (ω1 == ω2) && (And @@ ((∂h#σ1 == ∂h#σ2) & /@ S)) && (
    And @@ Flatten[Outer[
      (∂t#1h#2μ1 == ∂t#1h#2μ2) &,
      S, S
    ]]]
  )
]

```

```

Γ /: Γ[ω1_, σ1_, λ1_] Γ[ω2_, σ2_, λ2_] := Γ[ω1*ω2, σ1+σ2, λ1+λ2];
dmab→c[Γ[ω_, σ_, λ_]] := Module[{α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ},
  
$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a, h_a} \lambda & \partial_{t_a, h_b} \lambda & \partial_{t_a} \lambda \\ \partial_{t_b, h_a} \lambda & \partial_{t_b, h_b} \lambda & \partial_{t_b} \lambda \\ \partial_{h_a} \lambda & \partial_{h_b} \lambda & \lambda \end{pmatrix} / . (t | h)_{a|b} \rightarrow 0;$$

  ΓCollect[Γ[(μ = 1 - β) ω,
    hc (∂ha σ) (∂hb σ) + (σ / . ha|b → 0),
    {tc, 1} . (γ + α δ / μ ε + δ θ / μ) . {hc, 1}
    φ + α ψ / μ Ξ + θ ψ / μ] . {hc, 1}
  ] / . {Ta → Tc, Tb → Tc} // ΓCollect
];
dm[a_, b_, c_][Γ[ω_, σ_, λ_]] := dmab→c[Γ[ω, σ, λ]];
dη[a_][γr] := γ / . {(h | t)a → 0, Ta → 1};
FullStitch[γ1r, γ2r] := Module[{S1, S2, S, γ, τ},
  S = (S1 = dL[γ1]) ∪ (S2 = dL[γ2]);
  γ = γ1 (Times @@ (Γ / @ ε / @ Complement[S, S1]));
  γ *= (γ2 / . {ha → hτ[a], ta → tτ[a], Ta → Tτ[a]}
    (Times @@ (Γ / @ ε / @ τ / @ Complement[S, S2]));
  Do[
    γ = γ // dm[s, τ[s], s],
    {s, S}
  ];
  γ
];
Γ /: γ1r**γ2r := Module[{S1, S2, S, γ1p, γ2p},
  S = (S1 = dL[γ1]) ∪ (S2 = dL[γ2]);
  γ1p = γ1 (Times @@ (Γ / @ ε / @ Complement[S, S1]));
  γ2p = γ2 (Times @@ (Γ / @ ε / @ Complement[S, S2]));
  Γ[
    γ1p[ω] * γ2p[ω],
    (γ1p[Σ] γ2p[Σ]) . (h# & /@ S),
    (t# & /@ S) . (γ2p[A] . γ1p[A]) . (h# & /@ S)
  ]
];
Γ /: Γ[ω_, σ_, λ_]⁻¹ := Module[{S = dL[Γ[ω, σ, λ]]},
  Γ[
    ω⁻¹, Collect[σ, h_, (1/#) &],
    (t# & /@ S) . Inverse[Outer[ΓSimp[(∂ta1ha2 λ)] &, S, S]] . (h# & /@ S)
  ]
];

```



```

dA[a_][Γ[ω_, σ_, λ_]] := Module[
  {α, θ, φ, Ξ, σα},
  (α θ) = (∂_{t_a, h_a} λ ∂_{t_a} λ) /. (h | t)_a → 0;
  (φ Ξ) = (∂_{h_a} λ λ) /. (h | t)_a → 0;
  σα = ∂_{h_a} σ;
  rCollect[Γ[
    α ω / σα,
    ((σ /. h_a → 0) + h_a / σα),
    {t_a, 1} . (1 θ) . {h_a, 1} / α
  ]]]];
dS[a_][γ_F] := rCollect[dA[a][γ] /. T_a → 1/T_a];
Mirror[γ_F] := Module[{γ1},
  γ1 = γ // (dS@@dL[γ]);
  γ1[[3]] = γ1[[3]] /. {t_a_ → h_a, h_a_ → t_a};
  γ1];
tσ[rules__Rule][γ_F] := rCollect[
  γ /. {t_u_ → t_u /. {rules}, T_u_ → T_u /. {rules}}
];
hσ[rules__Rule][γ_F] := rCollect[γ /. h_x_ → h_x /. {rules}];
SetAttributes[Γ, Listable];
Γ[p_Times | p_NonCommutativeMultiply] := Γ /@ p;
Γ[e[a_]] := Γ[1, h_a, h_a t_a];
Γ[Xp[a_, b_]] := Γ[1, h_a + h_b T_a, {t_a, t_b} . (1 1 - T_a) . {h_a, h_b}];
Γ[Xm[a_, b_]] := Γ[Xp[a, b]] /. T_a → 1/T_a;
MVA[Γ, L_Link] := Module[{Hs, ω, σ, μ, A},
  {ω, σ, μ} = List @@ Z[Γ, L];
  Hs = Rest[h_# & /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h_a_ → t_a];
  Factor[
$$\frac{\omega \text{Det}[A - \text{IdentityMatrix}@\text{Length}@Hs]}{1 - \text{TSkeleton}[L][[1,1]}}$$
]]];

```

qΔ (“renormalized cabling”)

```

qΔ[a_, x_, y_] [Γ[ω_, σ_, λ_]] := Module [
  {α, θ, φ, Ξ, σα, Ta, M},
  ( α θ ) = ( ∂ta, ha λ ∂ta λ ) /. (h | t)a → 0 /. Ta → Ta;
  ( φ Ξ ) = ( ∂ha λ λ ) /. Ta → Ta;
  σα = ∂ha σ /. Ta → Ta;
  M = (
    ( -σα+α Ta+(-α+σα) Ty / -1+Ta, (-1+Tx) Ty (α-σα) / -1+Ta, θ (-1+Tx) Ty / -1+Ta )
    ( (-1+Ty) (α-σα) / -1+Ta, -α+σα Ta+(α-σα) Ty / -1+Ta, θ (-1+Ty) / -1+Ta )
    ( φ, φ, Ξ )
  );
  ΓCollect[Γ[
    ω /. Ta → Tx Ty,
    ((σ /. ha → 0) + (hx + hy) σα) /. Ta | Ta → Tx Ty,
    {tx, ty, 1}.M.{hx, hy, 1} /. Ta → Tx Ty
  ]
];

```

The Exact KV Solution in Γ

$$\Gamma[V] = \Gamma \left[\frac{\left(\frac{-1+T_1}{\text{Log}[T_1]} \right)^{1/4} \left(\frac{-1+T_2}{\text{Log}[T_2]} \right)^{1/4}}{\left(\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]} \right)^{1/4}}, h_1 + h_2 \sqrt{T_1}, \right.$$

$$\{t_1, t_2\} \cdot \left(\begin{array}{cc} \text{Log}[T_1] \left(1 + \frac{\text{Log}[T_2] \sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}}}{(-1+T_1) \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}} \right) & \text{Log}[T_1] \left(1 - \frac{\sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} T_2}{\sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}} \right) \\ \text{Log}[T_2] \left(1 - \frac{T_1 \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}}}{\sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}} \right) & \text{Log}[T_2] \left(1 + \frac{\text{Log}[T_1] \sqrt{\frac{(-1+T_1) T_1}{\text{Log}[T_1]}} T_2}{\text{Log}[T_2] \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}} \right) \end{array} \right) \cdot \{h_1, h_2\}$$

];

$$\Gamma[Vi] = \Gamma \left[\frac{\left(\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]} \right)^{1/4}}{\left(\frac{-1+T_1}{\text{Log}[T_1]} \right)^{1/4} \left(\frac{-1+T_2}{\text{Log}[T_2]} \right)^{1/4}}, h_1 + h_2 / \sqrt{T_1}, \right.$$

{t₁, t₂}.

$$\left(\begin{array}{cc} \frac{(-1+T_1) T_2 + \text{Log}[T_2] \sqrt{\frac{-1+T_1}{\text{Log}[T_1] T_1}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}{-1+T_1 T_2} & \frac{(-1+T_1) T_2 - \text{Log}[T_1] \sqrt{\frac{-1+T_1}{\text{Log}[T_1] T_1}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}{-1+T_1 T_2} \\ \frac{-1+T_2 - \text{Log}[T_2] \sqrt{\frac{-1+T_1}{\text{Log}[T_1] T_1}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}{-1+T_1 T_2} & \frac{-1+T_2 + \text{Log}[T_1] \sqrt{\frac{-1+T_1}{\text{Log}[T_1] T_1}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}{-1+T_1 T_2} \end{array} \right) \cdot$$

{h₁, h₂}

];

$\alpha \leftrightarrow \Gamma$ Conversions

```

A[ω_, μ_] // Γ := Module[{S},
  S = dL[{ω, μ}];
  Γ[ω,
    Total[h# & /@ S] + μ /. {h[a_] :=> ha, t[a_] :=> ca},
    Total[(t# h#) & /@ S] + (μ /. t[a_] :=> ca /. h[a_] :=> ha ta) - μ /.
      {h[a_] :=> ha, t[a_] :=> ca ta}
  ] /. ca :=> Log[Ta] // ΓCollect[FullSimplify[# /. Sinh[x_] :=>  $\frac{e^x - e^{-x}}{2}$ ] &]
];
    
```

```

Γ[ω_, σ_, λ_] // A := Module[{S, μ},
  S = dL[Γ[ω, σ, λ]];
  μ = Total[((∂h#σ) t#h#) & /@ S] - λ;
  A[ω,
    μ /. {ha -> h[a], ta -> t[a] / ca}
  ] /. Ta -> eca // αCollect
];

```

Γb-Calculus

```

ΓbSimp = Factor; SetAttributes[ΓbCollect, Listable];
ΓbCollect[Γb[ω_, σ_, λ_]] := ΓbCollect[ΓbSimp][Γb[ω, σ, λ]];
ΓbCollect[simp_][Γb[ω_, σ_, λ_]] := Γb[simp[ω], simp[σ],
  Collect[λ, h_, Collect[#, t_, simp] &]];
dL[Γb[_ , _ , λ_]] := dL[Γ[1, 1, λ]];
Γb[ω1_, _ , _][ω] := ω1;
Γb[ω_, σ_, λ_][Σ] := (∂h#σ) & /@ dL[Γb[ω, σ, λ]];
Γb[ω_, σ_, λ_][A] :=
  Module[{S = dL[Γb[ω, σ, λ]], Outer[ΓbSimp[(∂t#1h#2λ)] &, S, S]};
ΓbForm[Γb[ω_, σ_, λ_]] := Module[{S, M},
  S = dL[Γb[ω, σ, λ]];
  M = Γb[ω, σ, λ][A] // Transpose;
  PrependTo[M, s# & /@ S];
  M = Join[
    {Prepend[s# & /@ S, ω]},
    Transpose[M],
    {Prepend[Γb[ω, σ, λ][Σ], "Σ"]}
  ];
  MatrixForm[M]
];
ΓbForm[else_] := else /. Γb[ω_, σ_, λ_] -> ΓbForm[Γb[ω, σ, λ]];
Format[Γb[ω_, σ_, λ_], StandardForm] := ΓbForm[Γb[ω, σ, λ]];
Γ[ω_, σ_, λ_] // Γb := ΓbCollect[Γb[ω, σ, ω*λ]];
Γb[ω_, σ_, λ_] // Γ := ΓCollect[Γ[ω, σ, λ/ω]];
α_A // Γb := α // Γ // Γb

```

Γ -Calculus

```

Γ1Simp = Factor; SetAttributes[Γ1Collect, Listable];
Γ1Collect[Γ1[ω_, σ_, λ_]] := Γ1Collect[Γ1Simp][Γ1[ω, σ, λ]];
Γ1Collect[simp_][Γ1[ω_, σ_, λ_]] := Γ1[simp[ω], simp[σ],
  Collect[λ, h_, Collect[#, t_, simp] &]];
dL[Γ1[_ , _ , λ_]] := dL[Γ[1, 1, λ]];
Γ1[ω1_, _ , _][ω] := ω1;
Γ1[ω_, σ_, λ_][Σ] := (∂hsσ) & /@ dL[Γ1[ω, σ, λ]];
Γ1[ω_, σ_, λ_][A] :=
  Module[{S = dL[Γb[ω, σ, λ]], Outer[ΓbSimp[(∂ts1hs2λ)] &, S, S]};
Γ1Form[Γ1[ω_, σ_, λ_]] := Module[{S, M},
  S = dL[Γ1[ω, σ, λ]];
  M = Γ1[ω, σ, λ][A] // Transpose;
  PrependTo[M, s# & /@ S];
  M = Join[
    {Prepend[s# & /@ S, ω]},
    Transpose[M],
    {Prepend[Γ1[ω, σ, λ][Σ], "Γ1"}]
  ];
  MatrixForm[M]
];
Γ1Form[else_] := else /. Γ1[ω_, σ_, λ_] => Γ1Form[Γ1[ω, σ, λ]];
Format[Γ1[ω_, σ_, λ_], StandardForm] := Γ1Form[Γ1[ω, σ, λ]];

Γ[ω_, σ_, λ_] // Γ1 := Γ1Collect[Γ1[ω, σ, λ /. {hs -> (1 - Ts) hs, ts -> ts / (1 - Ts)}]];
Γ1[ω_, σ_, λ_] // Γ := ΓCollect[Γ[ω, σ, λ /. {hs -> hs / (1 - Ts), ts -> (1 - Ts) ts}}];
α_A // Γ1 := α // Γ // Γ1

SetAttributes[Γ1, Listable];
Γ1[p_Times | p_NonCommutativeMultiply] := Γ1 /@ p;
Γ1[e[a_]] := Γ1[1, ha, ha ta];
Γ1[Xp[a_, b_]] := Xp[a, b] // Γ // Γ1;
Γ1[Xm[a_, b_]] := Xm[a, b] // Γ // Γ1;

Γ1 /: Γ1[ω1_, σ1_, λ1_] Γ1[ω2_, σ2_, λ2_] := Γ1[ω1 * ω2, σ1 + σ2, λ1 + λ2];
γ_Γ1 // op_dm := γ // Γ // op // Γ1;
γ_Γ1 // op_qΔ := γ // Γ // op // Γ1;

```