Pensieve header: Studying closed components in Γ-calculus.

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/2014-07/MetaCalculi/"];
<< KnotTheory`
<< MetaCalculi-Program.m
ΓSimp = Assuming[T1 > 0 && T2 > 0 && T3 > 0 && T4 > 0 && T5 > 0 && T6 > 0, Simplify[#]] &
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

$$\text{Assuming}[T_1 > 0 \,\&\&\, T_2 > 0 \,\&\&\, T_3 > 0 \,\&\&\, T_4 > 0 \,\&\&\, T_5 > 0 \,\&\&\, T_6 > 0,\ \text{Simplify}[\#1]]\ \&$$

$$\text{tr}[a\_][\Gamma[\omega\_, \sigma\_, \lambda\_]] := \text{Module}\Big[\{\alpha, \theta, \psi, \Xi\},\ \begin{pmatrix} \alpha & \theta \\ \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a, h_a}\lambda & \partial_{t_a}\lambda \\ \partial_{h_a}\lambda & \lambda \end{pmatrix} /.\ (t \mid h)_a \to 0;$$
$$\Gamma\big[\omega\,(1-\alpha),\ \sigma\ /.\ h_a \to 0,\ \Xi + \psi * \theta \big/ (1-\alpha)\big] \ //\ \Gamma\text{Collect}\Big];$$

$$\text{TrB}[a\_][\Gamma[\omega\_, \sigma\_, \lambda\_]] := \text{Module}\Big[\{\alpha, \theta, \psi, \Xi\},\ \begin{pmatrix} \alpha & \theta \\ \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a, h_a}\lambda & \partial_{t_a}\lambda \\ \partial_{h_a}\lambda & \lambda \end{pmatrix} /.\ (t \mid h)_a \to 0;$$
$$\Gamma\big[\omega\,(1-\alpha),\ \sigma\ /.\ h_a \to 0,\ (1-\alpha)\,\Xi + \psi * \theta\big] \ //\ \Gamma\text{Collect}\Big];$$

Cyclicity of tr

$$n = 3;\ \gamma 0 = \Gamma\Big[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\Big]$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

```
{γ0 // dm[1, 2, 1], γ0 // dm[1, 2, 1] // tr[1]}
```

$$\left\{\begin{pmatrix} \omega - \omega\,\alpha_{12} & s_1 & s_3 \\ s_1 & \alpha_{21} + \frac{\alpha_{11}\,\alpha_{22}}{1-\alpha_{12}} & \frac{\alpha_{13}\,\alpha_{22}}{1-\alpha_{12}} + \alpha_{23} \\ s_3 & \alpha_{31} + \frac{\alpha_{11}\,\alpha_{32}}{1-\alpha_{12}} & \frac{\alpha_{13}\,\alpha_{32}}{1-\alpha_{12}} + \alpha_{33} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega\,(1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} - \alpha_{11}\,\alpha_{22}) & s_3 \\ s_3 & \frac{\alpha_{13}\,\alpha_{32}}{1-\alpha_{12}} + \frac{\left(\frac{\alpha_{13}\,\alpha_{22}}{1-\alpha_{12}} + \alpha_{23}\right)\left(\alpha_{31} + \frac{\alpha_{11}\,\alpha_{32}}{1-\alpha_{12}}\right)}{1 - \alpha_{21} + \frac{\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}}} + \alpha_{33} \\ \Gamma & \sigma_3 \end{pmatrix}\right\}$$

{γ0 // dm[2, 1, 1], γ0 // dm[2, 1, 1] // tr[1]}

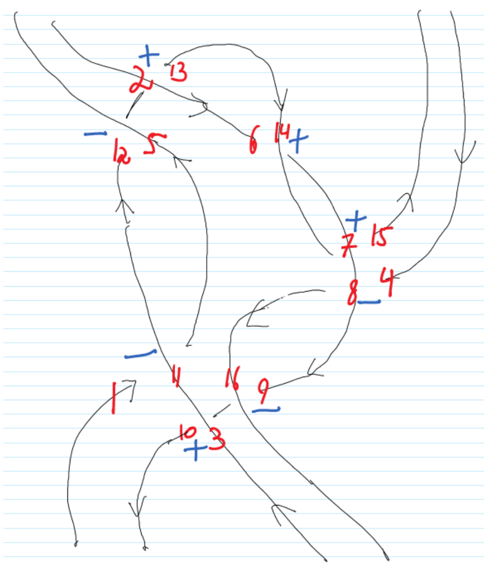
$$\left\{\begin{pmatrix} \omega - \omega\,\alpha_{21} & s_1 & s_3 \\ s_1 & \alpha_{12} + \frac{\alpha_{11}\,\alpha_{22}}{1-\alpha_{21}} & \alpha_{13} + \frac{\alpha_{11}\,\alpha_{23}}{1-\alpha_{21}} \\ s_3 & \frac{\alpha_{22}\,\alpha_{31}}{1-\alpha_{21}} + \alpha_{32} & \frac{\alpha_{23}\,\alpha_{31}}{1-\alpha_{21}} + \alpha_{33} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega\,(1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} - \alpha_{11}\,\alpha_{22}) & s_3 \\ s_3 & \frac{\alpha_{23}\,\alpha_{31}}{1-\alpha_{21}} + \frac{\left(\alpha_{13} + \frac{\alpha_{11}\,\alpha_{23}}{1-\alpha_{21}}\right)\left(\frac{\alpha_{22}\,\alpha_{31}}{1-\alpha_{21}} + \alpha_{32}\right)}{1-\alpha_{12} + \frac{\alpha_{11}\,\alpha_{22}}{-1+\alpha_{21}}} + \alpha_{33} \\ \Gamma & \sigma_3 \end{pmatrix}\right\}$$

$$\frac{\alpha_{13}\,\alpha_{32}}{1-\alpha_{12}} + \frac{\left(\frac{\alpha_{13}\,\alpha_{22}}{1-\alpha_{12}} + \alpha_{23}\right)\left(\alpha_{31} + \frac{\alpha_{11}\,\alpha_{32}}{1-\alpha_{12}}\right)}{1-\alpha_{21} + \frac{\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}}} + \alpha_{33} ==$$

$$\frac{\alpha_{23}\,\alpha_{31}}{1-\alpha_{21}} + \frac{\left(\alpha_{13} + \frac{\alpha_{11}\,\alpha_{23}}{1-\alpha_{21}}\right)\left(\frac{\alpha_{22}\,\alpha_{31}}{1-\alpha_{21}} + \alpha_{32}\right)}{1-\alpha_{12} + \frac{\alpha_{11}\,\alpha_{22}}{-1+\alpha_{21}}} + \alpha_{33} \text{ // Simplify}$$

True



$$\text{(*}\Gamma[\text{Xp}[a\_,b\_]] := \Gamma\left[\sqrt{T_a},\ h_a + h_b T_a,\ \{t_a, t_b\}.\begin{pmatrix} -\sqrt{T_a} & \frac{1-T_b}{\sqrt{T_a}} \\ 0 & -\frac{1}{\sqrt{T_a}} \end{pmatrix}.\{h_a, h_b\}\right];$$

$$\Gamma[\text{Xp}[a\_,b\_]] := \Gamma\left[\frac{1}{\sqrt{T_a}},\ h_a + h_b T_a,\ \{t_a, t_b\}.\begin{pmatrix} \frac{-1}{\sqrt{T_a}} & \frac{T_b-1}{\sqrt{T_a}} \\ 0 & -\sqrt{T_a} \end{pmatrix}.\{h_a, h_b\}\right];\text{*)}$$

R3

```
{Xm51 Xm62 Xp34 // Γ // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
  Xp61 Xm24 Xm35 // Γ // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

R3

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & 1-\frac{1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & \frac{1-T_3}{T_2} & 1-\frac{1}{T_3} & 1 \\ \Gamma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & 1-\frac{1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & \frac{1-T_3}{T_2} & 1-\frac{1}{T_3} & 1 \\ \Gamma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}\right\}$$

```
γ0 = Xm[11, 1] Xm[5, 12] Xp[2, 13] Xp[14, 6] Xp[7, 15] Xm[8, 4] Xm[16, 9] Xp[3, 10] // Γ;
γ1 =
 γ0 // dm[1, 5, 1] // dm[2, 6, 2] // dm[2, 7, 2] // dm[2, 8, 2] // dm[2, 9, 2] // dm[2, 10,
       2] // dm[3, 11, 3] // dm[3, 12, 3] //
    dm[3, 13, 3] // dm[3, 14, 3] // dm[3, 15, 3] // dm[4, 16, 4]
```

$$\begin{pmatrix} \frac{-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T_4}{T_1T_4} & s_1 & s_2 \\ s_1 & -\frac{1-T_2(-1+T_3)^2+T_2^2(-1+T_3)T_3-T_4+T_3(-1+T_4-T_1T_4)}{T_3\left(-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T_4\right)} & \frac{(1-T_1)(1+T_2(-1+T_3))(-1+T_3)}{-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T} \\ s_2 & \frac{(1-T_2)(-1+T_3)(1+T_2T_3)}{-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T_4} & \frac{T_1(1+T_2(-1+T_3))T_3}{-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T} \\ s_3 & \frac{T_2(-1+T_3)(1+(-1+T_2)T_3)T_4}{T_3\left(-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T_4\right)} & -\frac{T_2(-1+T_3)(T_2T_3+T_1T_4)}{-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1} \\ s_4 & \frac{(1-T_2)(-1+T_3)(1+T_2T_3)(-1+T_4)}{T_3\left(-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T_4\right)} & \frac{T_1(1+T_2(-1+T_3))(-1+T_4)}{-1+T_2(-1+T_3)^2+T_3-T_2^2(-1+T_3)T_3+T_1T} \\ \Gamma & \frac{1}{T_3} & \frac{T_3^2}{T_4} \end{pmatrix}$$

```
{γ1 // dm[1, 2, 1] // tr[1], γ1 // dm[2, 1, 1] // tr[1]}
```

$$\left\{\begin{pmatrix} \frac{-1-T_1(-1+T_3)+T_4}{T_4} & s_3 & s_4 \\ s_3 & \frac{T_1(T_3-T_4)}{1+T_1(-1+T_3)-T_4} & \frac{(-1+T_1)(-1+T_3)}{1+T_1(-1+T_3)-T_4} \\ s_4 & \frac{(-1+T_1)(-1+T_4)}{1+T_1(-1+T_3)-T_4} & \frac{T_3-T_4}{1+T_1(-1+T_3)-T_4} \\ \Gamma & T_1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1-\frac{1+T_1(-1+T_3)}{T_4} & s_3 & s_4 \\ s_3 & \frac{T_1(T_3-T_4)}{1+T_1(-1+T_3)-T_4} & \frac{(-1+T_1)(-1+T_3)}{1+T_1(-1+T_3)-T_4} \\ s_4 & \frac{(-1+T_1)(-1+T_4)}{1+T_1(-1+T_3)-T_4} & \frac{T_3-T_4}{1+T_1(-1+T_3)-T_4} \\ \Gamma & T_1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

$$1-\frac{1+T_1\left(-1+T_3\right)}{T_4} == \frac{-1-T_1\left(-1+T_3\right)+T_4}{T_4} \ \text{// Simplify}$$

True

```
{γ1 // dm[1, 2, 1] // TrB[1], γ1 // dm[2, 1, 1] // TrB[1]}
```

$$\left\{\begin{pmatrix} \frac{-1-T_1(-1+T_3)+T_4}{T_4} & s_3 & s_4 \\ s_3 & \frac{T_1(T_3-T_4)}{1+T_1(-1+T_3)-T_3-T_4} & \frac{(-1+T_1)(-1+T_3)}{1+T_1(-1+T_3)-T_3-T_4} \\ s_4 & \frac{(-1+T_1)(-1+T_4)}{1+T_1(-1+T_3)-T_3-T_4} & \frac{T_3-T_4}{1+T_1(-1+T_3)-T_3-T_4} \\ \Gamma & T_1 & \frac{1}{T_1} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1-\frac{1+T_1(-1+T_3)}{T_4} & s_3 & s_4 \\ s_3 & T_1\left(1-\frac{T_3}{T_4}\right) & -\frac{(-1+T_1)(-1+T_3)}{T_4} \\ s_4 & -\frac{(-1+T_1)(-1+T_4)}{T_4} & 1-\frac{T_3}{T_4} \\ \Gamma & T_1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

Testing the MVA

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\begin{pmatrix} \left(1-\left(1-\frac{1}{T_1}\right)\left(1-\frac{1}{T_5}\right)\left(1-\frac{1}{T_9}\right)\right)(1+(-1+T_1)(-1+T_5)(-1+T_9)) & s_1 \\ s_1 & \frac{(T_1(T_5(3-2T_9)+2(-1+T_9))+(-1+T_5)(-1+}{(T_1(-1+T_5)(-1+T_9)-T_5(-1+T_9)+T_9)((-1+} \\ s_5 & \frac{T_1(1+T_1(-1+T_5))(-1+T}{(T_1(-1+T_5)(-1+T_9)-T_5(-1+T_9)+T_9)((-1+} \\ s_9 & \frac{(-1+T_5)T_5(1+T_1(-2+T_9)-}{(1+(-1+T_1)(-1+T_5)(-1+T_9))((-1+T_5)} \\ \Gamma & 1 \end{pmatrix}$$

```
Z[Γ, Link["L6a4"]] // dL
```

{1, 5, 9}

```
Z[Γ, Link["L6a4"]] // tr[1] // tr[9]
```

$$\begin{pmatrix} \frac{(-1+T_1)(-1+T_5)^2(-1+T_9)}{T_5 T_9} & s_5 \\ s_5 & 1 \\ \Gamma & 1 \end{pmatrix}$$

```
trZ[L_] := Module[{γ},
  γ = Z[Γ, L];
  Do[
   γ = γ // tr[k],
   {k, Most[γ // dL]}];
  γ[ω] / (T_Last[γ//dL] - 1)]
```

```
trZ[Link["L10a4"]]
```

$$\frac{1}{T_5^2}(-1+T_1)(-1+T_5)\left(1-3T_5+3T_5^2-3T_5^3+T_5^4\right)$$

```
MultivariableAlexander[Link["L10a4"]][T]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$-\left(\left((-1+T[1])\,(-1+T[2])\,\left(1-3\,T[2]+3\,T[2]^2-3\,T[2]^3+T[2]^4\right)\right)\Big/\left(\sqrt{T[1]}\;T[2]^{5/2}\right)\right)$$

$$\texttt{Factor}\left[\frac{1}{\texttt{trZ[\#]}}\left(\texttt{MultivariableAlexander[\#][T] /. T[i\_]} \mapsto T_{\texttt{Skeleton[\#][[i,1]]}}\right)\right]\ \texttt{\& /@}$$

```
AllLinks[{2, 8}]
```

$$\Big\{-T_1^2\,T_3,\ -T_1^{3/2}\,T_5^{3/2},\ -\sqrt{T_1}\,T_5^{3/2},\ -T_1^{3/2}\,\sqrt{T_5},\ -T_1^2\,T_7^2,\ -T_1^2\,T_7^2,\ \frac{\sqrt{T_1}\,\sqrt{T_9}}{\sqrt{T_5}},\ T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2},$$
$$\frac{\sqrt{T_1}}{\sqrt{T_5}\,\sqrt{T_9}},\ -\frac{1}{\sqrt{T_1}\,\sqrt{T_5}},\ -T_1^{3/2}\,T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1\,T_7^2,\ -\frac{1}{T_7},\ \frac{T_1^{3/2}\,\sqrt{T_5}}{\sqrt{T_9}},\ -\sqrt{T_1}\,T_5^{5/2},$$
$$-\frac{T_5^{3/2}}{\sqrt{T_1}},\ -\frac{T_5^{5/2}}{\sqrt{T_1}},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}},\ -T_1^{3/2}\,T_5^{3/2},\ -\sqrt{T_1}\,T_5^{3/2},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -\sqrt{T_1}\,T_5^{9/2},\ -\frac{T_1}{T_7},$$
$$-T_1\,T_7,\ -T_1^2\,T_7^3,\ -T_1^2\,T_7^3,\ -T_1^{5/2}\,T_9^{5/2},\ -T_1^{5/2}\,T_9^{5/2},\ -T_1^{5/2}\,T_9^{5/2},\ T_1^{3/2}\,T_5^{3/2}\,\sqrt{T_9},\ \frac{1}{\sqrt{T_1}\,T_5},$$
$$T_1^{3/2}\,T_5^2\,T_{11}^2,\ \frac{\sqrt{T_1}}{T_5\,T_{11}},\ \sqrt{T_1},\ \frac{T_1^{3/2}}{\sqrt{T_5}\,\sqrt{T_{13}}},\ -T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\,T_{13}^{3/2},\ -\sqrt{T_1}\,T_5^{5/2},\ -\frac{T_5^{3/2}}{\sqrt{T_1}},$$
$$\sqrt{T_1}\,T_5\,T_{11}^2,\ \frac{T_5\,T_{11}}{\sqrt{T_1}},\ \frac{\sqrt{T_5}\,T_{11}^{3/2}}{\sqrt{T_1}},\ T_1^{3/2}\,T_5^{3/2}\,T_{13}^{3/2},\ -\frac{\sqrt{T_1}\,\sqrt{T_5}}{\sqrt{T_9}\,\sqrt{T_{13}}},\ -\sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_9}\,\sqrt{T_{13}}\Big\}$$

tr in Γbetter

$$\texttt{γ0 = Γ}\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,\ t_S\}.\begin{pmatrix}\alpha & \phi\\ \theta & \Xi\end{pmatrix}.\{h_a,\ h_S\}\right]$$

$$\begin{pmatrix}\omega & s_a & s_S\\ s_a & \alpha & \phi\\ s_S & \theta & \Xi\\ \Gamma & \sigma_a & \sigma_S\end{pmatrix}$$

```
((γ0 /. Thread[{α, ϕ, θ, Ξ} → {α, ϕ, θ, Ξ} / ω]) // tr[a])
```

$$\begin{pmatrix}-\alpha+\omega & s_S\\ s_S & \frac{\Xi-\frac{\theta\,\phi}{\alpha-\omega}}{\omega}\\ \Gamma & \sigma_S\end{pmatrix}$$

$$\Xi-\frac{\theta\,\phi}{\alpha-\omega}\ \texttt{// Simplify}$$

$$\Xi-\frac{\theta\,\phi}{\alpha-\omega}$$

Cyclicity of TrB?

```
n = 3; γ0 = Γ[ω, Sum[h_a σ_a, {a,0,n}], Sum[t_a h_b α_{10 a+b}, {a,1,n}, {b,1,n}]]
```

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

```
{γ0 // dm[1, 2, 1], γ0 // dm[1, 2, 1] // tr[1], γ0 // dm[1, 2, 1] // TrB[1]}
```

$$\left\{\begin{pmatrix} \omega - \omega\,\alpha_{12} & s_1 & s_3 \\ s_1 & \alpha_{21} + \frac{\alpha_{11}\,\alpha_{22}}{1-\alpha_{12}} & \frac{\alpha_{13}\,\alpha_{22}}{1-\alpha_{12}} + \alpha_{23} \\ s_3 & \alpha_{31} + \frac{\alpha_{11}\,\alpha_{32}}{1-\alpha_{12}} & \frac{\alpha_{13}\,\alpha_{32}}{1-\alpha_{12}} + \alpha_{33} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega\,(1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} - \alpha_{11}\,\alpha_{22}) & s_3 \\ s_3 & \frac{\alpha_{13}\,\alpha_{32}}{1-\alpha_{12}} + \frac{\left(\frac{\alpha_{13}\,\alpha_{22}}{1-\alpha_{12}}+\alpha_{23}\right)\left(\alpha_{31}+\frac{\alpha_{11}\,\alpha_{32}}{1-\alpha_{12}}\right)}{1-\alpha_{21}+\frac{\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}}} + \alpha_{33} \\ \Gamma & \sigma_3 \end{pmatrix}, \begin{pmatrix} \omega\,(1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} \\ s_3 \\ \Gamma \end{pmatrix}$$

```
{γ0 // dm[2, 1, 1], γ0 // dm[2, 1, 1] // tr[1], γ0 // dm[2, 1, 1] // TrB[1]}
```

$$\left\{\begin{pmatrix} \omega - \omega\,\alpha_{21} & s_1 & s_3 \\ s_1 & \alpha_{12} + \frac{\alpha_{11}\,\alpha_{22}}{1-\alpha_{21}} & \alpha_{13} + \frac{\alpha_{11}\,\alpha_{23}}{1-\alpha_{21}} \\ s_3 & \frac{\alpha_{22}\,\alpha_{31}}{1-\alpha_{21}} + \alpha_{32} & \frac{\alpha_{23}\,\alpha_{31}}{1-\alpha_{21}} + \alpha_{33} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega\,(1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} - \alpha_{11}\,\alpha_{22}) & s_3 \\ s_3 & \frac{\alpha_{23}\,\alpha_{31}}{1-\alpha_{21}} + \frac{\left(\alpha_{13}+\frac{\alpha_{11}\,\alpha_{23}}{1-\alpha_{21}}\right)\left(\frac{\alpha_{22}\,\alpha_{31}}{1-\alpha_{21}}+\alpha_{32}\right)}{1-\alpha_{12}+\frac{\alpha_{11}\,\alpha_{22}}{-1+\alpha_{21}}} + \alpha_{33} \\ \Gamma & \sigma_3 \end{pmatrix}, \begin{pmatrix} \omega\,(1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} \\ s_3 \\ \Gamma \end{pmatrix}$$

$$\left(\frac{1}{-1 + \alpha_{12}}\left(\alpha_{23}\,\left(\left(-1 + \alpha_{12}\right)\alpha_{31} - \alpha_{11}\,\alpha_{32}\right) - \alpha_{13}\,\left(\alpha_{22}\,\alpha_{31} - \left(-1 + \alpha_{21}\right)\alpha_{32}\right) + \left(-1 - \alpha_{12}\,\left(-1 + \alpha_{21}\right) + \alpha_{21} + \alpha_{11}\,\alpha_{22}\right)\alpha_{33}\right) == \frac{1}{-1 + \alpha_{21}}\left(\alpha_{23}\,\left(\left(-1 + \alpha_{12}\right)\alpha_{31} - \alpha_{11}\,\alpha_{32}\right) - \alpha_{13}\,\left(\alpha_{22}\,\alpha_{31} - \left(-1 + \alpha_{21}\right)\alpha_{32}\right) + \left(-1 - \alpha_{12}\,\left(-1 + \alpha_{21}\right) + \alpha_{21} + \alpha_{11}\,\alpha_{22}\right)\alpha_{33}\right)\right) \text{ // Simplify}$$

$$\frac{1}{(-1 + \alpha_{12})\,(-1 + \alpha_{21})}\,(\alpha_{12} - \alpha_{21})\,(\alpha_{23}\,(-(-1 + \alpha_{12})\,\alpha_{31} + \alpha_{11}\,\alpha_{32}) + \alpha_{13}\,(\alpha_{22}\,\alpha_{31} - (-1 + \alpha_{21})\,\alpha_{32}) + (1 + \alpha_{12}\,(-1 + \alpha_{21}) - \alpha_{21} - \alpha_{11}\,\alpha_{22})\,\alpha_{33}) == 0$$