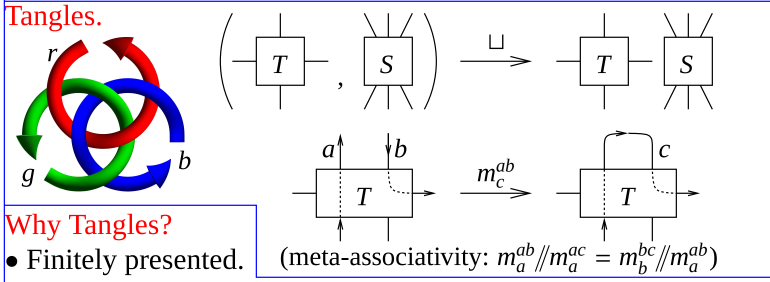
# Some very good formulas for the Alexander polynomial, 1

**Abstract.** I will describe some very good formulas for a *(matrix plus scalar)*-valued extension of the Alexander polynnomial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to *(free Lie algebras plus cyclic words)*, and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

## Tangles.



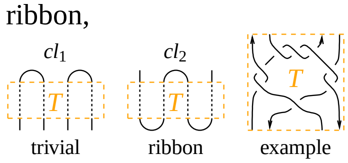

(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

## Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If $K$ is ribbon,

$Z(K) \in \{cl_2(Z) \colon cl_1(Z) = 1\}$.

(Genus and crossing number are also definable properties).




## Theorem 1.

$\exists!$ an invariant $\gamma$: {pure framed $S$-component tangles} $\to R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in $S$ variables, intertwining

**1.** $$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

**2.** $$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right)_{T_a,T_b\to T_c},$$

and satisfying $$\left(\uparrow_a;\ {}_a\!\nearrow\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\nearrow_a\right) \xrightarrow{\gamma} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array}; \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right).$$

## In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $L \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

## Implementation key idea:

ωεβ/Demo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
   M = Outer[Factor[∂h#1 t#2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ma_ b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ; ∂tb,ha λ  ∂tb,hb λ  ∂tb λ; ∂ha λ  ∂hb λ  λ) /. (t | h)a|b → 0;
  Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{hc, 1}]
    /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_ b_ := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rma_ b_ := Rpab /. Ta → 1 / Ta;
```

## Meta-Associativity

Runs.

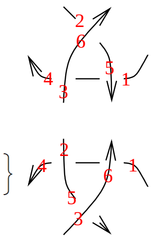
```
γ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(γ // m12→1 // m13→1) == (γ // m23→2 // m12→1)
True
```

### R3

... divide and conquer!

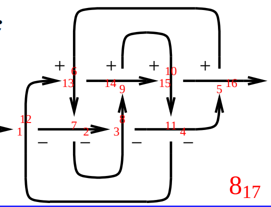
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
  Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$




```
γ = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[γ = γ // m1k→1, {k, 2, 16}];
γ
```

$$\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} & h_1\\ t_1 & 1\end{pmatrix}$$

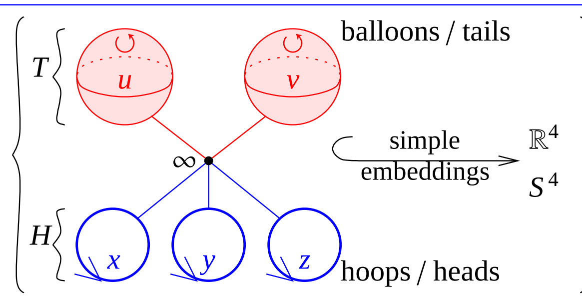



## Weaknesses,

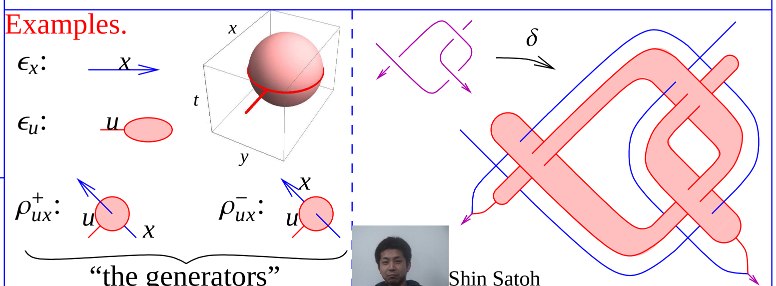
- $m^{ab}_c$ is non-linear.
- The product $\omega A$ is always Laurent, but proving this takes induction with exponentially many conditions.

## $\mathcal{K}^{bh}(H;T)$.

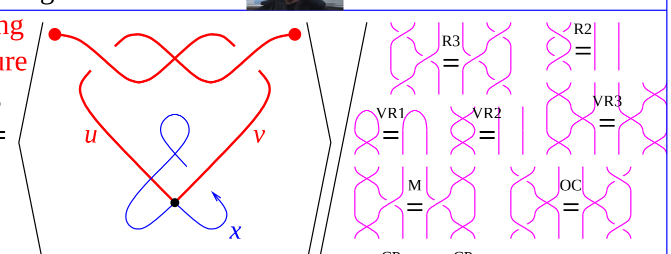
"Simply-knotted balloons and hoops"



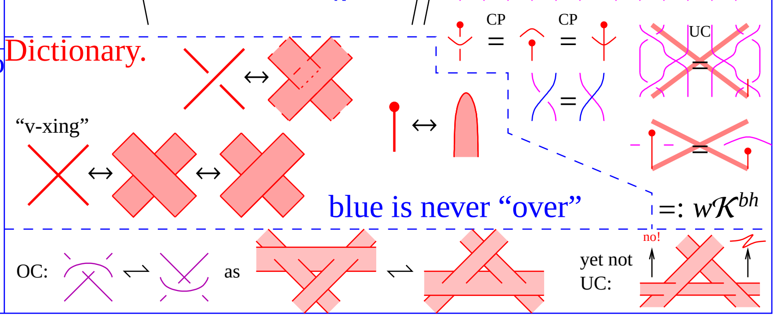

## Examples.




## Disturbing Conjecture

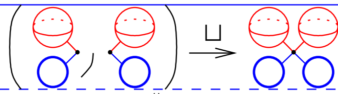



## Dictionary.



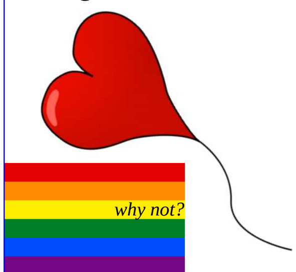
# Some very good formulas for the Alexander polynomial, 2

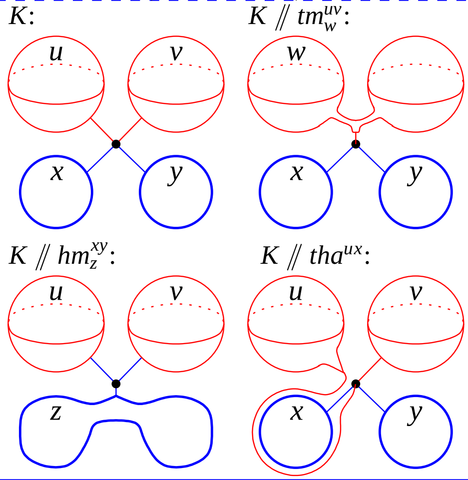
**Operations** Punctures & Cuts | Connected Sums.




If $X$ is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and $\pi_1$ acts on $\pi_2$.

**Proposition.** The generators generate.




**Definition.** $l_{xu}$ is the linking number of hoop $x$ with balloon $u$. For $x \in H$, $\sigma_x := \prod_{u\in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a\in T})$, the ring of rational functions in $T$ variables.

**Theorem 2** [BNS]. $\exists!$ an invariant $\beta\colon w\mathcal{K}^{bh}(H;T) \to R \times M_{T\times H}(R)$, intertwining

**1.** $$\left(\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array},$$

**2.** $$\begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \Xi\end{array} \xrightarrow{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \Xi\end{array}\right)_{T_u,T_v\to T_w},$$

**3.** $$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \Xi\end{array} \xrightarrow{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \Xi\end{array},$$

**4.** $$\begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \theta\\ T & \phi & \Xi\end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|cc}\nu\omega & x & H\\ \hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu\\ T & \phi/\nu & \Xi-\phi\theta/\nu\end{array},$$

and satisfying $(\epsilon_x;\epsilon_u;\rho^\pm_{ux}) \xrightarrow{\beta} \left(\begin{array}{c|c}1 & x\\ \hline & \end{array}; \begin{array}{c|c}1 & \\ \hline u & \end{array}; \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}\right)$.

**Proposition.** If $T$ is a u-tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \operatorname{diag}(\sigma_a)_{a\in S}$. Under this, $m^{ab}_c \leftrightarrow tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$.

**References.**

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

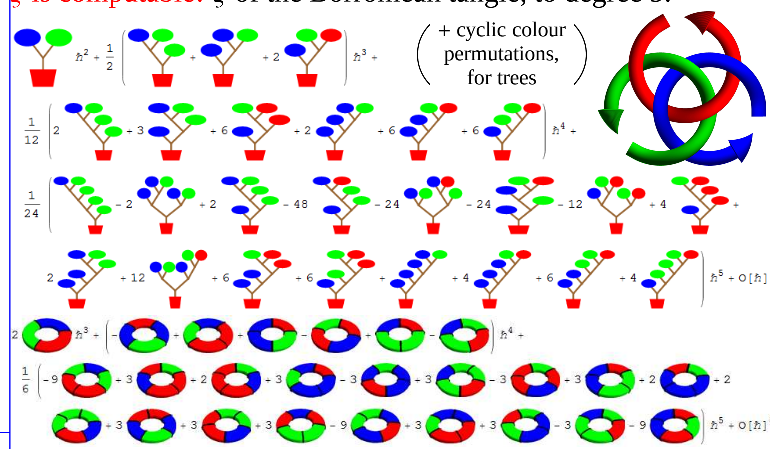
[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

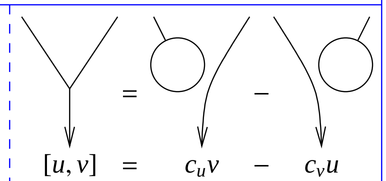
[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

**Theorem 3** [BND, BN]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant $Z$ of w-knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

**$\zeta$ is computable!** $\zeta$ of the Borromean tangle, to degree 5:

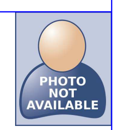



**Proposition** [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, $\zeta$ reduces to $\beta$ and the KBH operations on $\zeta$ reduce to the formulas in Theorem 2.




**A Big Question.** Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

**BF Following [CR].** $A \in \Omega^1(M=\mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,
$$S(A,B) := \int_M \langle B, F_A\rangle.$$

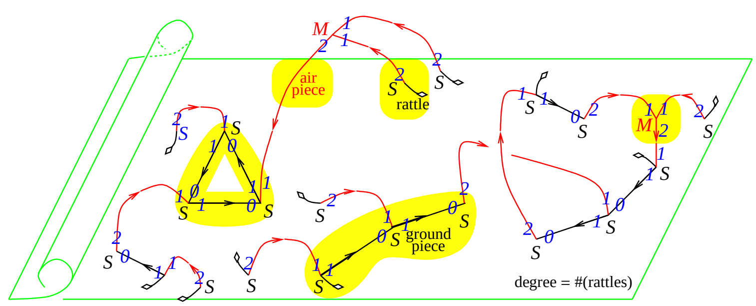


Rossi

With $\kappa\colon (S=\mathbb{R}^2) \to M$, $\beta \in \Omega^0(S,\mathfrak{g})$, $\alpha \in \Omega^1(S,\mathfrak{g}^*)$, set
$$\mathcal{O}(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle\beta, d_{\kappa^*A}\alpha + \kappa^*B\rangle\right).$$

Cattaneo

**The BF Feynman Rules.** For an edge $e$, let $\Phi_e$ be its direction, in $S^3$ or $S^1$. Let $\omega_3$ and $\omega_1$ be volume forms on $S^3$ and $S_1$. Then
$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$
(modulo some $STU$- and $IHX$-like relations).




**Issues.** • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant.

• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.

• I don't know how to define "finite type" for arbitrary 2-knots.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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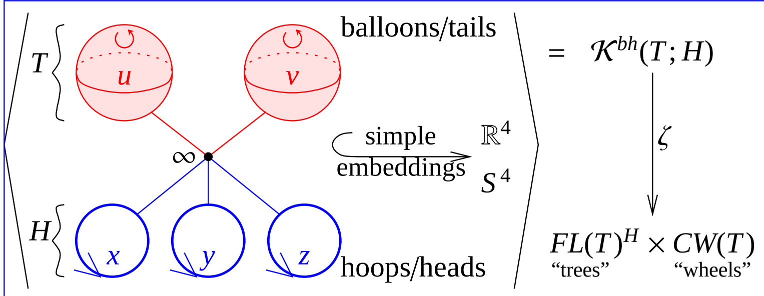
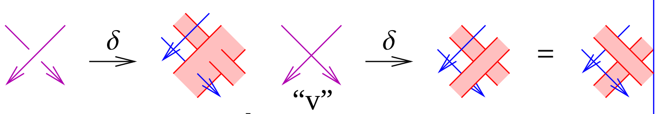
## Safekeeping / Recycling.




More on $\delta$




satisfies R123, VR123, D, and

- $\delta$ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- $\delta$ maps v-tangles to $\mathcal{K}^{bh}$; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, $\delta$ is onto.