BANFF WORKSHOP PROPOSAL:
KNOT THEORY AND VIRTUAL KNOT THEORY

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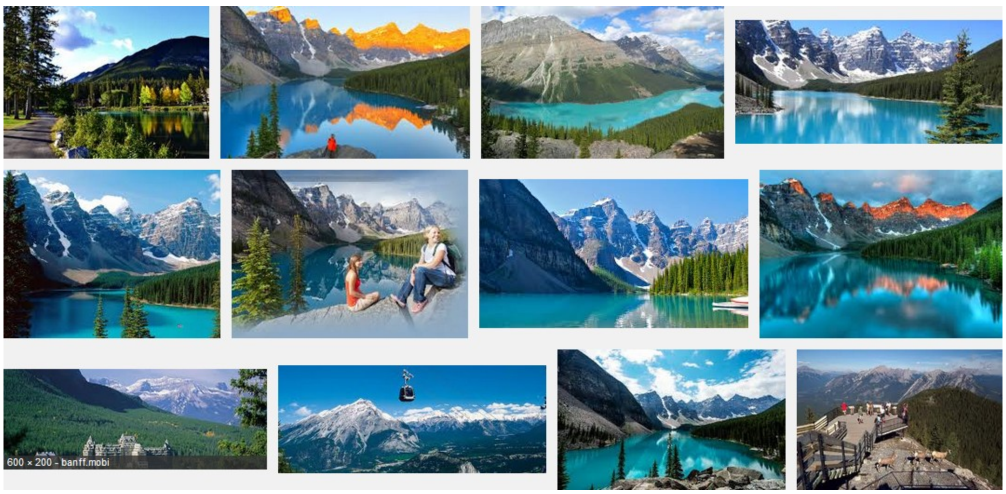

1. Overview. *A short overview of the subject area of your workshop.*

1.1. **Introduction.** It is by now well known, and vastly discussed and extended, that to every semi-simple Lie algebra $\mathfrak{g}$ and representation ρ thereof corresponds a knot invariant $Z_{\mathfrak{g},\rho}$. They can be defined in various ways; often the definition involves sophisticated algebra and the repeated uttering of phrases such as "quantum groups" and "Yang Baxter equations". $Z_{\mathfrak{g},\rho}$ is typically called "the quantum invariant corresponding to $\mathfrak{g}$ and ρ".

The primary topic of our proposed workshop is **being honest about the domain space of quantum invariants:** it isn't really knots, but "virtual knots". Virtual knots can be thought of as knots in thickened surfaces under an equivalence relation called "stabilization" (see ahead). The result is an object that is well-suited to combinatorial and topological investigations. The intrinsic combinatorial structures of virtual knots provide new insights into quantum invariants. The connections between these combinatorial insights and their algebraic and topological meaning for classical knot theory are just beginning to be developed.

Our proposed workshop will bring together researchers in classical knot theory, virtual knot theory, and the theory of knots in thickened surfaces. The goal is to build mathematical and

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interpersonal bridges between these communities. As the topic lends itself to such a vast diversity of approaches, the workshop will be accessible to young researchers and researchers at primarily undergraduate institutions.

1.2. **What are ordinary knots?** Often, ordinary knots are presented as pictures in the plane, and their (quantum) invariants can be computed from such pictorial presentations. Viewed as a planar object, a knot is an equivalence class of "knot diagrams", where a knot diagram is a planar quadrivalent graph with some mild further annotations (some orientations, plus at each vertex, or "crossing", an indication of which line is "over" and which is "under"). Such diagrams are equivalent if they differ only by "Reidemeister moves" that have very simple local descriptions. These are easy to find on, say, Wikipedia, but are hard to produce using the subset of LaTeX we are allowed to use here.

1.3. **What are virtual knots?** Following Kauffman [Ka1], these are exactly the same, except that the word "planar" is dropped. The annotated quadrivalent graphs that make knot diagrams make sense just as well without being embedded in the plane. The Reidemeister moves also carry over. Some additional relations are added to account for the absence of planarity. In some algebraic sense, virtual knots are more natural than ordinary knots. Indeed, graphs have various simple algebraic descriptions, and using any of these, it is also easy to describe the Reidemeister moves. But planarity is a global property; when graphs are converted to algebra, planarity typically becomes hard to describe.

Back in topology, virtual knots can be regarded as equivalence classes of knots embedded in a "thickened surface" (the cartesian product of a short interval with a surface, potentially of a high genus), modulo "stabilization" — the addition or removal of empty handles, a move that ensures that the virtual knot remains an abstract quantity, not associated with any specific surface. Using this image and a little about the 3-dimensional topology of thickened surfaces, Kuperberg [Ku] showed that virtual knots have unique "minimal genus models", and that the obvious map of ordinary knots into virtual knots is an injection, and hence virtual knots theory is strictly a superset of ordinary knot theory.

1.4. **What are Quantum Invariants?** Ignoring mild issues having to do with "caps and cups", and, only in order to simplify this presentation, ignoring the difference between matrices and their inverses, the formulas for quantum invariants can be read directly from the annotated quadrivalent graphs describing knot diagrams. Namely, the theory of quantum groups associates a 4-index $N \times N \times N \times N$ tensor R^{ab}_{cd} to each $(\mathfrak{g}, \rho)$ pair, "the R-matrix". Now given a quadrivalent graph, mark its edges with variables a, b, etc., and note that exactly four letters appear around each vertex of the graph. Take one copy of the tensor R for each vertex of the graph and index it with the four variables that surround it. Multiply all these R-matrices and sum each variable running from 1 to N. The result (forgiving our few white lies) is $Z_{\mathfrak{g},\rho}$.

For example, applying the above procedure to the two sides of the Reidemeister 3 move (strictly speaking, these are "tangles" and not "knots"), we get

$$\sum_{d,e,f=1}^{N} R^{ab}_{de} R^{ec}_{fi} R^{df}_{gh} \qquad \text{and} \qquad \sum_{d,e,f=1}^{N} R^{bc}_{de} R^{ad}_{gf} R^{fe}_{hi}$$

The equality of the above two formulas is the celebrated "Yang-Baxter Equation", which is satisfied by R-matrices that come from quantum groups. Hence quantum groups allow us to associate a knot invariant to any $(\mathfrak{g}, \rho)$ pair.

But note that planarity was not even mentioned above! Hence the more natural domain of definition of quantum invariants is the more natural realm of "virtual knots". When all the white lies are untold, the conclusion remains the same (with a slightly modified meaning for "virtual knots", but the distinction is immaterial here).

The consequences of the extension to virtual knots can be surprising. Consider the Jones polynomial. There are non-classical virtual knots that have unit Jones polynomial. It is unknown if the Jones polynomial is an unknot detector for classical knots. To settle it, one need only prove that all virtual knots having unit Jones polynomial are non-classical.

2. Objectives. *A statement of the objectives of the workshop and an indication of its relevance, importance, and timeliness.*

Our official cause for celebration is the 20th anniversary of Kauffman's first discussion of virtual knot theory in 1996 (belatedly published at [Ka1]). Our true objective is the good old true and tested: we'd like to bring together the experts in the field, from both the "virtual" and the "quantum" side, along with some younger researchers (meaning, the future experts) in the hope that the great things that have been done will be communicated and that the great things of the next few years will be initiated. Recent research has suggested areas that we believe deserve immediate attention. Two such areas will be focused on at the workshop.

2.1. **Finite type invariants and Virtual Knots.** Finite type invariants are numerical invariants defined on their domain spaces (be it knots, or virtual knots, or any of many other kinds objects) which, in some sense, behave polynomially as a function of their inputs. In calculus, a "polynomial" is something that vanishes when differentiated enough times. Alternatively, derivatives can be replaced by differences of values at neighboring points (without taking limits), and polynomials can be defined to be "those functions whose iterated differences vanish". The latter notion makes sense at a much higher generality — all that one needs is a notion of "neighboring points" for which iteration makes sense. For ordinary knots, this turns out to be "knots that differ at only one crossing", and this leads to an elegant theory of polynomial, or more technically "finite type" invariants of knots. While the definitions are very simple, the construction of the full set of finite type invariants is difficult. It can be achieved either using the so called "Kontsevich integral", or using Feynman diagram techniques that arise from the Chern-Simons-Witten topological quantum field theory, or by a delicate use of Drinfel'd's theory of quasi-Hopf algebras (the theory of "associators"). In fact, we believe that the best way to understand Drinfel'd's theory of associators is as the algebraic construction of a "homomorphic universal finite type invariant" of an appropriate class of (ordinary) knotted objects.

Note that the "neighborhood" of a knot within virtual knots is not the same as within ordinary knots, and hence the notion of finite type invariants (or polynomial invariants) of virtual knots is different from the corresponding notion for ordinary knots (and not merely by a restriction). Hence it matters that the natural domain of quantum invariants is virtual knots; it means that the appropriate finite-type context for quantum invariants is the virtual one. In fact, it appears that just as finite type invariants of ordinary knots are related to the Drinfel'd theory of associators, so is the finite type theory of virtual knots related to the quantization of Lie bi-algebras and of solutions of the classical Yang-Baxter equation [EK, En]. We believe that these quantization problems will ultimately be commonly phrased as problems about the construction of finite type invariants of virtual knots.

2.2. **Bridges to Low-Dimensional Topology.** Previously virtual knots have been discussed as a generalization of knots in 3-space. This is advantageous from an algebraic perspective. A natural question is whether virtual knots provide any insight into the topology and geometry of classical knots. The answer to this question is "yes"! Two such bridges to low-dimensional topology are described below.

First, Satoh showed [Sat] that there is a certain "suspension" map δ that turns virtual knots into 2-dimensional knots in $\mathbb{R}^4$ and that all "simply knotted 2-knots" are in the image of δ (the kernel is conjectured but not proven). Thus, virtual knots describe a part of the theory of 2-knots.

Second, Chrisman and Manturov [CM, C] have shown (recently) that multi-component links in 3-space and knots in 3-manifolds can be studied with virtual knots. The technique is called the theory of virtual covers. Take for example a knot K in a fibered link complement N. The theory of virtual covers associates a virtual knot υ to K. In many cases, υ turns out to be an ambient isotopy invariant of K in N. Geometric properties of K in N are encoded in the virtual knot υ. Moreover, these properties can be detected using virtual knot invariants. Thus, virtual knots also describe a part of the theory of links in 3-space and knots in 3-manifolds.

The existence of these bridges suggests the following general avenue of investigation. Take a problem in low-dimensional topology. Apply a "bridge" to turn it into a problem about virtual knots. Then one can take advantage of the intrinsic combinatorial structures in virtual knot theory in order to understand the original problem. The previous successes suggest that there are many such bridges to low-dimensional topology. We would like to assemble the experts on each side of the divide so that more bridges can be constructed. Additionally, we would like to better understand the consequences of the existing bridges for low-dimensional topology.

2.3. **There's more!** In order to start from the very basics and yet reach some depth within a small space, we've had to cut a few corners and omit a great deal. Knots come along with long knots, links, braids, tangles, knotted graphs, and with higher dimensional analogs. The same is true for virtual knots. There is much active research about all these "virtually knotted objects". Many topics in ordinary knot theory extend to virtual knots, often with some extra punch: the Alexander polynomial [Saw, BDGGHN] (with extra punch!), the Kauffman bracket [Ka2, DKM] (extra punch!), the fundamental group [BB] (extra punch!), the Artin, Burau, and Gassner representations, categorification (sometimes), and more.

Some aspects of virtual knot theory are completely new. Just as relaxing the condition of planarity gave rise to virtual knots, relaxing the over-under crossing structure gives rise to the non-trivial theory of *flat virtual knots* [Ch]. Similarly, flat virtual knots can be further simplified by introducing equivalence up to the "virtualization move". This produces the theory of *free knots*. Central to this multiverse of knot theories is *parity* [Ma]. It provides an powerful method by which to enhance quantum invariants of virtual knots.

In short, there is much to do. The proposed Banff workshop will establish the ground work for these new directions in knot theory and virtual knot theory.

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3. Press Release. *Please provide 1-2 paragraphs for a press release for your workshop. It should be understandable by the general public.*

If the natural numbers 1,2,3,... are the simplest and purest of algebra, knots are the simplest and purest of "topology" — that part of mathematics which studies what cannot be changed by a gentle morphing, and can only be modified by abruptly cutting and tearing. Knots are bent and bending pieces of string in our usual 3-dimensional space. Sailors and mountain climbers have been using them since ever, and mathematicians have been studying them for almost as long.

The topic of our workshop is "virtual knots". A relatively recent generalization of the classical notion of knots, which sometimes represents ordinary knots and sometimes represents knots in 4-dimensional space, yet in general lives in no dimension in particular. One reason we care about virtual knots is because of their deep relationship with algebra in general and with "quantum algebra" in particular.

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