BANFF PROPOSED PROPOSAL: KNOT THEORY AND VIRTUAL KNOT THEORY

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To my co-organizers. Good or bad, the below is how I see things. To me, virtual knots are mostly a vehicle to study quantum invariants, and that's how I wrote things. At the end perhaps it looks more like DBN's research proposal, rather than a plan for a conference on a thriving and diverse field. But I wrote what I know how to write, and I'm not sure how to do it better. If somebody else wants to add or subtract or rewrite, feel free. As always, you may fire me, or else, if the proposal at the end is too far from my own feelings, I may resign. I'd like to emphasize that even if I am fired or if I resign, you may use any part of my proposed proposal or even the whole thing as you please.

I'm particularly unhappy about part 2, "objectives". That part calls for insincerity and exaggeration, and I've already spent all my available insincerity and exaggeration in writing part 1.

Perhaps we should hold a video conference sometime early next week to talk about this some more. I think it is important to bring the written proposal to good shape very soon. This will allow us to show it to potential invitees so they can evaluate whether or not they are seriously interested. The more serious people are seriously interested the better our chances.

— DBN.

1. Overview. *A short overview of the subject area of your workshop.*

It is by now well known, and vastly discussed and extended, that to every semi-simple Lie algebra $\mathfrak{g}$ and representation ρ thereof corresponds a knot invariant $Z_{\mathfrak{g},\rho}$ which can be defined in various ways which often involve sophisticated algebra and the repeated uttering of phrases such as "quantum groups" and "Yang Baxter equations". $Z_{\mathfrak{g},\rho}$ is often called "the quantum invariant corresponding to $\mathfrak{g}$ and ρ".

The primary topic of our proposed workshop is **being honest about the domain space of quantum invariants:** it isn't really knots, but "virtual knots". An explanation is required, and a case should be made that this deserves study. Both appear in the next few paragraphs.

What are ordinary knots? Often, ordinary knots are presented as pictures in the plane, and their (quantum) invariants can be computed from such ~~a~~ pictorial presentation. Viewed as a planar object, a knot is an equivalence class of "knot diagrams", where a knot diagram is a planar quadrivalent graph with some mild further annotations (some orientations, plus at each vertex, or "crossing", an indication of which line is "over" and which is "under").

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The full TEX sources are at http://drorbn.net/AcademicPensieve/2014-07/BanffProposal/.

Following [Ka], they are exactly

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Such diagrams are equivalent if they differ only by "Reidemeister moves" that have very simple local descriptions ~~[~~these are easy to find on, say, Wikipedia, but are hard to produce using the subset of LaTeX we are allowed to use here~~]~~.

What are virtual knots? ~~Exactly~~ the same, except that the word "planar" is dropped. The annotated quadrivalent graphs that make knot diagrams make sense just as well without being embedded in the plane, and so do the Reidemeister moves. In some algebraic sense, virtual knots are more natural than ordinary knots. Indeed, graphs have various simple algebraic descriptions, and using any of these, it is easy to also describe the Reidemeister moves. But planarity is a global property; when graphs are converted to algebra, planarity typically becomes hard to describe.

Back in topology, virtual knots can be regarded as equivalence classes of knots embedded in a "thickened surface" (the cartesian product of a short interval with a surface, potentially of a high genus), modulo "stabilization" — addition and removal of empty handles, a move that ensures that the virtual knot remains an abstract quantity, not associated with any specific surface. Using this image and a little about the 3-dimensional topology of thickened surfaces, Kuperberg [] had shown that virtual knots have unique "minimal genus models", and that the obvious map of ordinary knots into virtual knots is an injection, and hence virtual knots theory is strictly a superset of ordinary knot theory. The

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Later, Satoh [] found that there is a certain "suspension" map δ that turns virtual knots into 2-dimensional knots in $\mathbb{R}^4$ and that all "simply knotted 2-knots" are in the image of δ (the kernel is conjectured but not proven). Thus virtual knots describe a part of the theory of 2-knots.

Quantum Invariants. Ignoring mild issues having to do with "caps and cups", and, only in order to simplify this presentation, ignoring the difference between matrices and their inverses, the formulas for quantum invariants can be read directly from the annotated quadrivalent graphs describing knot diagrams. Namely, the theory of quantum groups associates a 4-index $N \times N \times N \times N$ tensor R^{ab}_{cd} to each $(\mathfrak{g}, \rho)$ pair, "the R-matrix". Now given a quadrivalent graph, mark its edges with variables a, b, etc., and note that exactly four letters appear around each vertex of the graph. Take one copy of the tensor R for each vertex of the graph and index it with the four variables that surround it. Multiply all these R-matrices and sum each variable running from 1 to N. The result (forgiving our few white lies) is $Z_{\mathfrak{g},\rho}$.

For example, applying the above procedure to the two sides of the Reidemeister 3 move (strictly speaking, these are "tangles" and not "knots", ~~but the distinction is immaterial~~), we get

$$\sum_{d,e,f=1}^{N} R^{ab}_{de} R^{ec}_{fi} R^{df}_{gh} \qquad \text{and} \qquad \sum_{d,e,f=1}^{N} R^{bc}_{de} R^{ad}_{gf} R^{fe}_{hi}$$

The equality of the above two formulas is the celebrated "Yang-Baxter Equation", which is satisfied by R-matrices that come from quantum groups. Hence quantum groups allow us to associate a knot invariant to any $(\mathfrak{g}, \rho)$ pair.

But note that planarity was not even mentioned above! Hence the more natural domain of definition of quantum invariants is the more natural realm of "virtual knots". When all the white lies are untold, the conclusion remains the same, ~~albeit~~ (with a slightly modified meaning for "virtual knots"), but the distinction is immaterial here)

Why should we study virtual knots? The philosophical answer is that in math we tend to study anything that is natural, and virtual knots are natural: they arise both as a natural generalization of ordinary knots, and (slightly modified) as the natural domain of the theory of quantum invariants. A more concrete rationale occurs within the theory of finite type invariants.

What are finite type invariants? They are numerical invariants defined on their domain spaces (be it knots, or virtual knots, or any of many other kinds objects) which, in some sense, behave polynomially as a function of their inputs. In calculus, a "polynomial" is something that vanishes ~~when~~ differentiated enough times. Alternatively, derivatives can be of values replaced by differences at neighboring points (without taking limits), and polynomials can be defined to be "those functions whose iterated differences vanish". The latter notion makes sense at a much higher generality — all that one needs is a notion of "neighboring points" for which iteration makes sense. For ordinary knots, this turns out to be "knots that differ at only one crossing", and this leads to an elegant theory of polynomial, or more technically "finite type" invariants of knots. While the definitions are very simple, the construction of the full set of finite type invariants of is difficult. It can be achieved either using the so called "Kontsevich integral", or using Feynman diagram techniques that arise from the Chern-Simons-Witten topological quantum field theory, or by a delicate use of Drinfel'd's theory of quasi-Hopf algebras (the theory of "associators"). In fact, we believe that the best ~~clearest~~ way to understand Drinfel'd's theory of associators is as the algebraic construction of a "homomorphic universal finite type invariant" of an appropriate class of ~~(classically)~~ ordinary knotted objects.

So why virtuals? Note that the "neighborhood" of a knot within virtual knots is not the same as within ordinary knots, and hence the notion of finite type invariants, or polynomial invariants, of virtual knots is different from the corresponding notion for ordinary knots (and not merely by a restriction). Hence it matters that the natural domain of quantum invariants is virtual knots; it means that the appropriate finite-type context for quantum invariants is the virtual one. In fact, it appears that just as finite type invariants of ordinary knots are related to the Drinfel'd theory of associators, ~~so does~~ the finite type theory of virtual knots is ~~is~~ related to the quantization of Lie bi-algebras [EK, En-ij.] and of solutions of the classical Yang-Baxter equation. We believe that these quantization problems will ~~ultimately~~ be commonly phrased as problems about the construction of finite type invariants of virtual knots.

There's more! In order to start from the very basics and yet reach some depth within a short overview, we've had to cut a few corners and omit a great deal. Knots come along with links and braids and tangles and knotted graphs, and with higher dimensional analogs. The same is true for virtual knots, and there is much active research about all these "virtually knotted objects". Many topics in ordinary knot theory extend to virtual knots, often with some extra punch: the Alexander polynomial [Sw, Bo] (with extra punch!), the Kauffman bracket [Ka, Dye] (extra punch!), the Burau and Gassner representations, categorification (sometimes), and more. And some aspects of virtual knot theory are completely new [parities].

2. Objectives.

A statement of the objectives of the workshop and an indication of its relevance, importance, and timeliness.

Our official cause for celebration is the 20th anniversary of Kauffman's first discussion of virtual knot theory in 1996 (belatedly published at [Ka]). Our true objective is the good old true and tested: we'd like to bring together the experts in the field, from both the "virtual" and the "quantum" side, along with some younger researchers (meaning, the future experts) in the hope that the great things that have been done will be communicated and that the great things of the next few years will be initiated.

References

[Ka] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

3. Press Release. *Please provide 1-2 paragraphs for a press release for your workshop. It should be understandable by the general public.*

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If the natural numbers 1,2,3,... are the simplest and purest of algebra, knots are the simplest and purest of "topology" — that part of mathematics which studies what cannot be changed by a gentle morphing, and can only be modified by abruptly cutting and tearing. Knots are bent and bending pieces of string in our usual 3-dimensional space, sailors and mountain climbers have been using them since ever, and mathematicians have been studying them for almost as long.

The topic of our workshop is "virtual knots". A relatively recent generalization of the classical notion of knots, which sometimes represents ordinary knots and sometimes represents knots in 4-dimensional space, yet in general lives in no dimension in particular. One reason we care about virtual knots is because of their deep relationship with algebra in general and with "quantum algebra" in particular.

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