Dror Bar-Natan: Academic Pensieve: 2014-06:

A \$100 Problem on the Gassner Representation

June-24-14 8:34 AM

June 30 update: problem solved! See http://drorbn.net/AcademicPensieve/2014-06/UnitarityOfGassner/

Let t and t_i, $1\leq i\leq n$ be formal variables. Denote $\bar{t}=t^{-1}$, $\bar{t}_i=t_i^{-1}$ (makes sense if t, t_i represent complex numbers of unit norm).

Let $U_i(t)$ be the $n\times n$ matrix $U_i(t)=\begin{pmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 1-t & t & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & I_{n-i-1}\end{pmatrix}$;

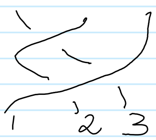
its inverse is $U_i^{-1}(t)=\begin{pmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \bar{t} & 1-\bar{t} & 0\\ 0 & 0 & 0 & I_{n-i-1}\end{pmatrix}$.

Let $b\in uB_n$ be a braid; $b=\prod_{\alpha=1}^{k}\sigma_{i_\alpha}^{s_\alpha}$; and let j_α denote the number of the under-strand in crossing $\#\alpha$ in b, as counted at the bottom of the braid.

The Gassner invariant $\Gamma(b)(t_1,\ldots t_n)$ is the product

$$\Gamma(b):=\prod_{\alpha=1}^{k}U_{i_\alpha}^{s_\alpha}(t_{j_\alpha})\in M_{n\times n}(\overbrace{\mathbb{Z}[t_i,t_i^{-1}]}^{R})$$

Example & verification of R3:



$=\sigma_1\sigma_2\sigma_1\mapsto U_1(t_2)U_2(t_3)U_1(t_3)$

Academic Pensieve: 2014-06: MiniGassner.nb:

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In[1]:= U[n_, i_, t_] := ReplacePart[
          IdentityMatrix[n],
          {{i, i} → 1 - t, {i, i + 1} → t,
           {i + 1, i} → 1, {i + 1, i + 1} → 0}
        ];

In[2]:= MatrixForm /@ (ms1 = {U[3, 1, t2], U[3, 2, t3], U[3, 1, t3]})
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Out[2]= $\left\{\begin{pmatrix} 1-t_2 & t_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_3 & t_3 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_3 & t_3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$

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In[3]:= Dot @@ ms1 // Simplify // MatrixForm
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Out[3]//MatrixForm=

$$\begin{pmatrix} 1-t_3 & -(-1+t_2)\,t_3 & t_2\,t_3 \\ 1-t_3 & t_3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \sigma_2\sigma_1\sigma_2 \mapsto U_2(t_3)\,U_1(t_3)\,U_2(t_2)$$

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In[4]:= MatrixForm /@ (ms2 = {U[3, 2, t3], U[3, 1, t3], U[3, 2, t2]})
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Out[4]= $\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_3 & t_3 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_3 & t_3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_2 & t_2 \\ 0 & 1 & 0 \end{pmatrix}\right\}$

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In[5]:= Dot @@ ms2 // Simplify // MatrixForm
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Out[5]//MatrixForm=

$$\begin{pmatrix} 1-t_3 & -(-1+t_2)\,t_3 & t_2\,t_3 \\ 1-t_3 & t_3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Problem. Write $\Gamma(b) = (\gamma_{ij})$. Find an explicit algebraic formula for $\bar{\gamma}_{ij}$ in terms of the γ_{ij}'s, with coefficients in R.

Example/exercise. The Burau representation is obtained by identifying all the variables in the Gassner invariant to a single one, denoted t. Let U be the result:

$$U(b) = \prod_{\alpha=1}^{k} U_{i_\alpha}^{s_\alpha}(t) = \Gamma(b) \,//\, (t_i \to t),$$

Then U satisfies $\bar{U}\Omega U^T = \Omega$, where

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ 1-t & 1 & 0 & 0 & \cdots \\ 1-t & 1-t & 1 & 0 & \\ 1-t & 1-t & 1-t & 1 & \\ & \vdots & & & \ddots \end{pmatrix}$$

If you solve the problem and show me your solution by Friday July 4, 2014 at noon, and if you are the first to do so, I owe you $100. Your solution must really be fully explicit, and I retain the right to decide what that means.

I don't expect this problem to be terribly hard. A solution is probably written down in one of the references below, except I'm too confused to be able to translate it to my language. Good Luck!

You may wish to look back at the web version of this page for corrections, modifications, and updates.

Dror.

Some References.

1. Abdulrahim, "A Faithfulness Criterion for the Gassner Representation of the Pure Braid Group".
2. Kassel-Turaev, "Braid Groups" (book).
3. Kirk-Livingston-Wang "The Gassner Representation for String Links".
4. D.D. Long, "On the Linear Representations of Braid Groups".
5. C.C. Squier, "The Burau Representation is Unitary".