Pensieve Header: The depth-mirror property in Gassner calculus.

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/2014-06/"];
<< KnotTheory`
<< "MetaCalculi/MetaCalculi-Program.m"
rSimp = Factor;
```

```
Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
V // A
```

$$\left(\begin{array}{ccc}
\frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\
t[1] & \frac{-\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1 c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\
t[2] & \frac{\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}c_1\sqrt{\quad}}{\quad}
\end{array}\right.$$

```
V1 = V // A /. c_a_ :> -c_a
```

$$\left(\begin{array}{ccc}
\frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\
t[1] & \frac{-\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1 c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\
t[2] & \frac{\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}c_1\sqrt{\quad}}{\quad}
\end{array}\right.$$

```
V // Γ
```

$$\left(\begin{array}{ccc}
\frac{\left(\frac{-1+T_1}{\mathrm{Log}[T_1]}\right)^{1/4}\left(\frac{-1+T_2}{\mathrm{Log}[T_2]}\right)^{1/4}}{\left(\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}\right)^{1/4}} & s_1 & s_2 \\
s_1 & \frac{\mathrm{Log}[T_1]\left(\mathrm{Log}[T_2]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}-\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}+T_1\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}\right)}{\mathrm{Log}[T_1T_2](-1+T_1)\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}} & -\frac{\mathrm{Log}[T_1]\left(\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\,T_2-\sqrt{\frac{-1+\cdots}{\mathrm{Log}[\cdots}}\cdots\right.}{\mathrm{Log}[T_1T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\cdots}} \\
s_2 & \frac{\mathrm{Log}[T_2]\left(-T_1\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}+\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}\right)}{\mathrm{Log}[T_1T_2]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}}} & \frac{\mathrm{Log}[T_1]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\,T_2+\mathrm{Log}[T_2]\sqrt{\cdots}}{\mathrm{Log}[T_1T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\cdots}} \\
\Sigma & 1 & \sqrt{T_1}
\end{array}\right.$$

```
V1 // Γ
```

$$\left(\begin{array}{cc}
\frac{\left(\frac{-1+T_1}{\mathrm{Log}[T_1]\sqrt{T_1}}\right)^{1/4}\left(\frac{-1+T_2}{\mathrm{Log}[T_2]\sqrt{T_2}}\right)^{1/4}}{\left(\frac{-1+T_1T_2}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])\sqrt{T_1}\sqrt{T_2}}\right)^{1/4}} & \\
s_1 & \frac{\mathrm{Log}[T_1]\left(\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]\sqrt{T_2}}}-T_1\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]\sqrt{T_2}}}-T_1\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]\sqrt{T_2}}}\,T_2+T_1^2\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]\sqrt{T_2}}}\,T_2-\mathrm{Log}[T_1]\sqrt{\frac{\cdots}{\mathrm{Log}[T\cdots}}\cdots\right.}{\cdots} \\
s_2 & \\
\Sigma &
\end{array}\right.$$

```
V1 == (V // A // dA[All] // dS[All]) // FullSimplify
 (* result is mess for Simplify, did not finish for FullSimplify *)
```

## The Garside Element

```
G[1] = ϵ[1] // Γ;
G[n_] /; n > 1 := G[n] = (G[n - 1] // qΔ[n - 1, n - 1, n]) ** (Xp[n - 1, n] // Γ);
Gi[n_] := Gi[n] = G[n]^-1;
```

```
{G[1], G[2], Xp[1, 2] // Γ}
```

$$\left\{\left(\begin{array}{cc} 1 & s_1 \\ s_1 & 1 \\ \Sigma & 1 \end{array}\right),\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right),\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right)\right\}$$

```
{G[3], Xp[1, 2] ** Xp[1, 3] ** Xp[2, 3] // Γ, G[3] ** Gi[3]}
```

$$\left\{\left(\begin{array}{cccc} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 1-T_1 & 1-T_1 \\ s_2 & 0 & T_1 & -T_1(-1+T_2) \\ s_3 & 0 & 0 & T_1T_2 \\ \Sigma & 1 & T_1 & T_1T_2 \end{array}\right),\left(\begin{array}{cccc} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 1-T_1 & 1-T_1 \\ s_2 & 0 & T_1 & -T_1(-1+T_2) \\ s_3 & 0 & 0 & T_1T_2 \\ \Sigma & 1 & T_1 & T_1T_2 \end{array}\right),\left(\begin{array}{cccc} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & 0 \\ s_2 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 1 \\ \Sigma & 1 & 1 & 1 \end{array}\right)\right\}$$

```
{G[4], G[5]}
```

| 1 | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $s_1$ | 1 | $1-T_1$ | $1-T_1$ | $1-T_1$ |
| $s_2$ | 0 | $T_1$ | $-T_1(-1+T_2)$ | $-T_1(-1+T_2)$ |
| $s_3$ | 0 | 0 | $T_1 T_2$ | $-T_1 T_2(-1+T_3)$ |
| $s_4$ | 0 | 0 | 0 | $T_1 T_2 T_3$ |
| $\Sigma$ | 1 | $T_1$ | $T_1 T_2$ | $T_1 T_2 T_3$ |

| 1 | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $s_1$ | 1 | $1-T_1$ | $1-T_1$ | $1-T_1$ | $1-T_1$ |
| $s_2$ | 0 | $T_1$ | $-T_1(-1+T_2)$ | $-T_1(-1+T_2)$ | $-T_1(-1+T_2)$ |
| $s_3$ | 0 | 0 | $T_1 T_2$ | $-T_1 T_2(-1+T_3)$ | $-T_1 T_2(-1+T_3)$ |
| $s_4$ | 0 | 0 | 0 | $T_1 T_2 T_3$ | $-T_1 T_2 T_3(-1+T_4)$ |
| $s_5$ | 0 | 0 | 0 | 0 | $T_1 T_2 T_3 T_4$ |
| $\Sigma$ | 1 | $T_1$ | $T_1 T_2$ | $T_1 T_2 T_3$ | $T_1 T_2 T_3 T_4$ |

```
{Gi[4], Gi[5]}
```

| 1 | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $s_1$ | 1 | $\frac{-1+T_1}{T_1}$ | $\frac{-1+T_1}{T_1}$ | $\frac{-1+T_1}{T_1}$ |
| $s_2$ | 0 | $\frac{1}{T_1}$ | $\frac{-1+T_2}{T_1 T_2}$ | $\frac{-1+T_2}{T_1 T_2}$ |
| $s_3$ | 0 | 0 | $\frac{1}{T_1 T_2}$ | $\frac{-1+T_3}{T_1 T_2 T_3}$ |
| $s_4$ | 0 | 0 | 0 | $\frac{1}{T_1 T_2 T_3}$ |
| $\Sigma$ | 1 | $\frac{1}{T_1}$ | $\frac{1}{T_1 T_2}$ | $\frac{1}{T_1 T_2 T_3}$ |

| 1 | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $s_1$ | 1 | $\frac{-1+T_1}{T_1}$ | $\frac{-1+T_1}{T_1}$ | $\frac{-1+T_1}{T_1}$ | $\frac{-1+T_1}{T_1}$ |
| $s_2$ | 0 | $\frac{1}{T_1}$ | $\frac{-1+T_2}{T_1 T_2}$ | $\frac{-1+T_2}{T_1 T_2}$ | $\frac{-1+T_2}{T_1 T_2}$ |
| $s_3$ | 0 | 0 | $\frac{1}{T_1 T_2}$ | $\frac{-1+T_3}{T_1 T_2 T_3}$ | $\frac{-1+T_3}{T_1 T_2 T_3}$ |
| $s_4$ | 0 | 0 | 0 | $\frac{1}{T_1 T_2 T_3}$ | $\frac{-1+T_4}{T_1 T_2 T_3 T_4}$ |
| $s_5$ | 0 | 0 | 0 | 0 | $\frac{1}{T_1 T_2 T_3 T_4}$ |
| $\Sigma$ | 1 | $\frac{1}{T_1}$ | $\frac{1}{T_1 T_2}$ | $\frac{1}{T_1 T_2 T_3}$ | $\frac{1}{T_1 T_2 T_3 T_4}$ |

```
{Xp[1, 2] ϵ[3] // Γ, G[3] ** (Xp[2, 1] // Γ) ** (Gi[3] // dσ[2, 1])}
```

| 1 | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $s_1$ | 1 | $1-T_1$ | 0 |
| $s_2$ | 0 | $T_1$ | 0 |
| $s_3$ | 0 | 0 | 1 |
| $\Sigma$ | 1 | $T_1$ | 1 |

| 1 | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $s_1$ | 1 | $1-T_1$ | 0 |
| $s_2$ | 0 | $T_1$ | 0 |
| $s_3$ | 0 | 0 | 1 |
| $\Sigma$ | 1 | $T_1$ | 1 |

```
{Xp[2, 3] ϵ[1] // Γ, G[3] ** (Xp[3, 2] // Γ) ** (Gi[3] // dσ[1, 3, 2])}
```

| 1 | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $s_1$ | 1 | 0 | 0 |
| $s_2$ | 0 | 1 | $1-T_2$ |
| $s_3$ | 0 | 0 | $T_2$ |
| $\Sigma$ | 1 | 1 | $T_2$ |

| 1 | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $s_1$ | 1 | 0 | 0 |
| $s_2$ | 0 | 1 | $1-T_2$ |
| $s_3$ | 0 | 0 | $T_2$ |
| $\Sigma$ | 1 | 1 | $T_2$ |

```
{t1 = Xp[1, 2] ** Xm[3, 1] ** Xp[2, 3] ** Xm[1, 2] ** Xp[3, 1] ** Xm[2, 3] // Γ,
  t2 = G[3] ** (t3 =
      Xp[2, 1] ** Xm[1, 3] ** Xp[3, 2] ** Xm[2, 1] ** Xp[1, 3] ** Xm[3, 2] // Γ) ** Gi[3],
  t1 == t2 // Simplify} // ColumnForm
```

| 1 | $s_1$ | $s_2$ |
|---|---|---|
| $s_1$ | $-\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1}$ | $\frac{(-1+T_1)\,(1-T_2+T_1 T_2)\,(-1}{T_1}$ |
| $s_2$ | $\frac{(-1+T_2)\,(-1+T_3)\,(-1+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3)}{T_1 T_2 T_3}$ | $-\frac{1-T_1-T_2+T_1 T_2-2 T_3+2 T_1 T_3+3 T_2 T_3-5 T_1 T_2 T_3+T_1^2 T_2 T_3-T_2^2 T_3+2 T_1 T_2^2 T_3-T_1^2 T_2^2 T_3+}{T_1 T_2 T_3}$ |
| $s_3$ | $\frac{(-1+T_2)\,(-1+T_3)\,(1-T_3+T_1 T_3)}{T_1 T_2 T_3}$ | $-\frac{(-1+T_1)\,(-1+T_3)\,(-1+T_2+T_3-T_2}{T_1 T_2 T_3}$ |
| Σ | 1 | 1 |


| 1 | $s_1$ | $s_2$ |
|---|---|---|
| $s_1$ | $-\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1}$ | $\frac{(-1+T_1)\,(1-T_2+T_1 T_2)\,(-1}{T_1}$ |
| $s_2$ | $\frac{(-1+T_2)\,(-1+T_3)\,(-1+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3)}{T_1 T_2 T_3}$ | $-\frac{1-T_1-T_2+T_1 T_2-2 T_3+2 T_1 T_3+3 T_2 T_3-5 T_1 T_2 T_3+T_1^2 T_2 T_3-T_2^2 T_3+2 T_1 T_2^2 T_3-T_1^2 T_2^2 T_3+}{T_1 T_2 T_3}$ |
| $s_3$ | $\frac{(-1+T_2)\,(-1+T_3)\,(1-T_3+T_1 T_3)}{T_1 T_2 T_3}$ | $-\frac{(-1+T_1)\,(-1+T_3)\,(-1+T_2+T_3-T_2}{T_1 T_2 T_3}$ |
| Σ | 1 | 1 |


True

```
(t1 // dA[All]) ** t2
```

| 1 | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $s_1$ | 1 | 0 | 0 |
| $s_2$ | 0 | 1 | 0 |
| $s_3$ | 0 | 0 | 1 |
| Σ | 1 | 1 | 1 |

```
t1
```


| 1 | $s_1$ | $s_2$ |
|---|---|---|
| $s_1$ | $-\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1}$ | $\frac{(-1+T_1)\,(1-T_2+T_1 T_2)\,(-1}{T_1}$ |
| $s_2$ | $\frac{(-1+T_2)\,(-1+T_3)\,(-1+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3)}{T_1 T_2 T_3}$ | $-\frac{1-T_1-T_2+T_1 T_2-2 T_3+2 T_1 T_3+3 T_2 T_3-5 T_1 T_2 T_3+T_1^2 T_2 T_3-T_2^2 T_3+2 T_1 T_2^2 T_3-T_1^2 T_2^2 T_3+}{T_1 T_2 T_3}$ |
| $s_3$ | $\frac{(-1+T_2)\,(-1+T_3)\,(1-T_3+T_1 T_3)}{T_1 T_2 T_3}$ | $-\frac{(-1+T_1)\,(-1+T_3)\,(-1+T_2+T_3-T_2}{T_1 T_2 T_3}$ |
| Σ | 1 | 1 |


```
t3
```

| 1 | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|
| $s_1$ | $2-T_1-T_2+T_1 T_2-T_3+T_1 T_3+T_2 T_3-T_1 T_2 T_3$ | $(-1+T_1)\,(-1+T_3)$ | $-\frac{(-1+T_1)\,(-1+T_2)}{T_1 T_2}$ |
| $s_2$ | $-\frac{(-1+T_2)\,(-1+T_3)}{T_3}$ | 1 | $\frac{(-1+T_1)\,(-1+T_2)}{T_1 T_2 T_3}$ |
| $s_3$ | $\frac{(-1+T_2)\,(-1+T_3)\,(1-T_3+T_1 T_3)}{T_3}$ | $-(-1+T_1)\,(-1+T_3)$ | $\frac{-1+T_1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+2 T_1 T}{T_1 T_2 T_3}$ |
| Σ | 1 | 1 | 1 |

```
t3 // dS[All] // dA[All]
```

$$\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{-1+T_1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+2 T_1 T_2 T_3}{T_1 T_2 T_3} & \frac{(-1+T_1)(-1+T_3)}{T_1 T_3} & -(-1+T_1)(-1+T_2) \\ s_2 & -\frac{(-1+T_2)(-1+T_3)}{T_2} & 1 & (-1+T_1)(-1+T_2) T_3 \\ s_3 & \frac{(-1+T_2)(-1+T_3)(1-T_1+T_1 T_3)}{T_1 T_2 T_3} & -\frac{(-1+T_1)(-1+T_3)}{T_1 T_3} & 2-T_1-T_2+T_1 T_2-T_3+T_1 T_3+T_2 T_3-T_1 T_2 T_3 \\ \Sigma & 1 & 1 & 1 \end{pmatrix}$$

```
Thread[((t1[A] // Transpose // Flatten) /. T_ → T) ==
   Flatten[(t3 // dS[All])[A] /. T_ → T]] // Simplify
```

$$\left\{\text{True}, \frac{1}{T}+3T+T^3==2+3T^2, \frac{(-1+T)(1+T^3)}{T}==0, 2+T^2==\frac{1}{T}+2T, \text{True}, \frac{1}{T}+3T+T^3==3+2T^2, T==\frac{1}{T}, 2+T^2==\frac{1}{T}+2T, \text{True}\right\}$$

```
(t3 // dS[All])[A]
```

$$\left\{\left\{-\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1}, \frac{(-1+T_1)(-1+T_3)(-1+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3)}{T_1 T_3}, \frac{(-1+T_1)(-1+T_2)(1-T_3+T_1 T_3)}{T_1}\right\},\right.$$
$$\left\{\frac{(-1+T_2)(1-T_2+T_1 T_2)(-1+T_3)}{T_1 T_2}, -\frac{1}{T_1 T_2 T_3}\left(1-T_1-T_2+T_1 T_2-2T_3+2T_1 T_3+3T_2 T_3-5T_1 T_2 T_3+T_1^2 T_2 T_3-T_2^2 T_3+2T_1 T_2^2 T_3-T_1^2 T_2^2 T_3+T_3^2-T_1 T_3^2-2T_2 T_3^2+3T_1 T_2 T_3^2-T_1^2 T_2 T_3^2+T_2^2 T_3^2-2T_1 T_2^2 T_3^2+T_1^2 T_2^2 T_3^2\right),\right.$$
$$\left.-\frac{(-1+T_1)(-1+T_2)(-1+T_2+T_3-T_2 T_3+T_1 T_2 T_3)}{T_1 T_2}\right\}, \left\{-\frac{(-1+T_2)(-1+T_3)}{T_1 T_2}, \frac{(-1+T_1)(-1+T_3)(1-T_3+T_2 T_3)}{T_1 T_2 T_3}, \frac{-1+T_1+T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_2}\right\}\right\}$$

```
{n = 4; γ0 = Γ[ω, Sum[h_a σ_a, {a,0,n}], Sum[Sum[t_a h_b α_ab, {a,1,n}], {b,1,n}]], G[4] ** γ0 ** Gi[4] == γ0 // Simplify}
```

$$\left\{\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ s_2 & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ s_3 & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ s_4 & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}, \frac{(-1+T_1)(\alpha_{2,1}+\alpha_{3,1}+\alpha_{4,1})}{T_1}==0 \;\&\&\right.$$

$$\frac{1}{T_1}(-1+T_1)\left(-\alpha_{2,1}-\alpha_{3,1}-\alpha_{4,1}+T_1(\alpha_{1,1}-\alpha_{1,2}+\alpha_{2,1}-\alpha_{2,2}+\alpha_{3,1}-\alpha_{3,2}+\alpha_{4,1}-\alpha_{4,2})\right)==0 \;\&\&$$

$$\alpha_{1,1}+T_1(\alpha_{1,2}+T_2\,\alpha_{1,3}) ==$$
$$T_1\alpha_{1,1}+T_1 T_2\alpha_{1,2}+\alpha_{1,3}+\frac{(-1+T_1)\left((-1+T_1)\alpha_{2,1}+T_1((-1+T_2)\alpha_{2,2}-T_2\alpha_{2,3})\right)}{T_1}+$$
$$\frac{(-1+T_1)\left((-1+T_1)\alpha_{3,1}+T_1((-1+T_2)\alpha_{3,2}-T_2\alpha_{3,3})\right)}{T_1}+$$

$$\frac{(-1+T_1)\ ((-1+T_1)\ \alpha_{4,1}+T_1\ ((-1+T_2)\ \alpha_{4,2}-T_2\ \alpha_{4,3}))}{T_1}\ \&\&$$

$$\alpha_{1,1}+T_1\ (\alpha_{1,2}+T_2\ (\alpha_{1,3}+T_3\ \alpha_{1,4}))==T_1\ \alpha_{1,1}+T_1\ T_2\ \alpha_{1,2}+T_1\ T_2\ T_3\ \alpha_{1,3}+\alpha_{1,4}+\frac{1}{T_1}$$
$$(-1+T_1)\ ((-1+T_1)\ \alpha_{2,1}+T_1\ ((-1+T_2)\ \alpha_{2,2}+T_2\ ((-1+T_3)\ \alpha_{2,3}-T_3\ \alpha_{2,4})))+\frac{1}{T_1}$$
$$(-1+T_1)\ ((-1+T_1)\ \alpha_{3,1}+T_1\ ((-1+T_2)\ \alpha_{3,2}+T_2\ ((-1+T_3)\ \alpha_{3,3}-T_3\ \alpha_{3,4})))+\frac{1}{T_1}$$
$$(-1+T_1)\ ((-1+T_1)\ \alpha_{4,1}+T_1\ ((-1+T_2)\ \alpha_{4,2}+T_2\ ((-1+T_3)\ \alpha_{4,3}-T_3\ \alpha_{4,4})))\ \&\&$$

$$\left(-1+\frac{1}{T_1}\right)\alpha_{2,1}+\frac{(-1+T_2)\ (\alpha_{3,1}+\alpha_{4,1})}{T_1\ T_2}==0\ \&\&$$

$$\left(-1+\frac{1}{T_1}\right)\alpha_{2,1}==\frac{(-1+T_2)\ ((-1+T_1)\ \alpha_{3,1}-\alpha_{4,1}-T_1\ (\alpha_{3,2}-\alpha_{4,1}+\alpha_{4,2}))}{T_1\ T_2}\ \&\&$$

$$\frac{1}{T_1\ T_2}\Big(( -1+T_1)\ \alpha_{3,1}-T_1\ \alpha_{3,2}-\alpha_{4,1}+T_1\ \alpha_{4,1}-T_1\ \alpha_{4,2}+$$
$$T_2\ (-(-1+T_1)\ \alpha_{2,1}+\alpha_{3,1}+\alpha_{4,1}+T_1\ (\alpha_{2,2}-\alpha_{3,1}+2\ \alpha_{3,2}-\alpha_{3,3}-\alpha_{4,1}+2\ \alpha_{4,2}-\alpha_{4,3}))+$$
$$T_1\ T_2^2\ (-\alpha_{2,2}+\alpha_{2,3}-\alpha_{3,2}+\alpha_{3,3}-\alpha_{4,2}+\alpha_{4,3})\Big)==\alpha_{2,3}\ \&\&\ \frac{1}{T_1\ T_2}$$
$$\Big((-1+T_1)\ \alpha_{3,1}-T_1\ \alpha_{3,2}-\alpha_{4,1}+T_1\ \alpha_{4,1}-T_1\ \alpha_{4,2}+T_1\ T_2^2\ (-\alpha_{2,2}-(-1+T_3)\ \alpha_{2,3}+T_3\ \alpha_{2,4}-\alpha_{3,2}+$$
$$\alpha_{3,3}-T_3\ \alpha_{3,3}+T_3\ \alpha_{3,4}-\alpha_{4,2}+\alpha_{4,3}-T_3\ \alpha_{4,3}+T_3\ \alpha_{4,4})+T_2\ (-(-1+T_1)\ \alpha_{2,1}+\alpha_{3,1}+\alpha_{4,1}+$$
$$T_1\ (\alpha_{2,2}-\alpha_{3,1}+2\ \alpha_{3,2}-\alpha_{3,3}+T_3\ \alpha_{3,3}-T_3\ \alpha_{3,4}-\alpha_{4,1}+2\ \alpha_{4,2}-\alpha_{4,3}+T_3\ \alpha_{4,3}-T_3\ \alpha_{4,4}))\Big)==$$

$$\alpha_{2,4}\ \&\&\ \frac{-\alpha_{4,1}+T_3\ (\alpha_{3,1}+\alpha_{4,1})}{T_1\ T_2\ T_3}==\alpha_{3,1}\ \&\&\ \frac{1}{T_1\ T_2\ T_3}((-1+T_1)\ \alpha_{4,1}-T_1\ \alpha_{4,2}+$$
$$T_3\ (-(-1+T_1)\ \alpha_{3,1}+\alpha_{4,1}+T_1\ (\alpha_{3,2}-\alpha_{4,1}+\alpha_{4,2})))==\alpha_{3,2}\ \&\&$$

$$\frac{1}{T_1\ T_2\ T_3}(-(-1+T_1)\ \alpha_{4,1}+T_1\ (-(-1+T_2)\ \alpha_{4,2}+T_2\ \alpha_{4,3})+$$
$$T_3\ ((-1+T_1)\ \alpha_{3,1}-\alpha_{4,1}+T_1\ ((-1+T_2)\ \alpha_{3,2}+\alpha_{4,1}-\alpha_{4,2}+T_2\ \alpha_{4,2}-T_2\ \alpha_{4,3})))==0\ \&\&$$

$$\frac{1}{T_1\ T_2\ T_3}\Big((-1+T_1)\ \alpha_{4,1}+T_1\ ((-1+T_2)\ \alpha_{4,2}-T_2\ \alpha_{4,3})+$$
$$T_1\ T_2\ T_3^2\ (-\alpha_{3,3}+\alpha_{3,4}-\alpha_{4,3}+\alpha_{4,4})+T_3\ (-(-1+T_1)\ \alpha_{3,1}+\alpha_{4,1}-$$
$$T_1\ ((-1+T_2)\ \alpha_{3,2}+\alpha_{4,1}-\alpha_{4,2}+T_2\ (-\alpha_{3,3}+\alpha_{3,4}+\alpha_{4,2}-2\ \alpha_{4,3}+\alpha_{4,4})))\Big)==0\ \&\&$$

$$\frac{\alpha_{4,1}}{T_1\ T_2\ T_3}==\alpha_{4,1}\ \&\&\ \frac{-(-1+T_1)\ \alpha_{4,1}+T_1\ \alpha_{4,2}}{T_1\ T_2\ T_3}==\alpha_{4,2}\ \&\&$$
$$\frac{(-1+T_1)\ \alpha_{4,1}+T_1\ ((-1+T_2)\ \alpha_{4,2}+T_2\ (-1+T_3)\ \alpha_{4,3})}{T_1\ T_2\ T_3}==0\Big\}$$

```
t1 = V // Γ // dA[All] // dS[All]
```

$$\begin{pmatrix} -\frac{\left(\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1}\right)^{1/4}\left(\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right] T_2}\right)^{1/4}(-1+T_1 T_2)}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2 \left(\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}\right)^{5/4}} & s_1 \\ s_1 & -\frac{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1 T_2 \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}} \left(-\mathrm{Log}\left[\frac{1}{T_2}\right] \sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\, T_1 \sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right] T_2}} - \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}} + T_1 \cdots\right.}{(-1+T_1)(-1+T_1 T_2)} \\ s_2 & -\frac{\mathrm{Log}\left[\frac{1}{T_2}\right] T_2 \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}} \left(-\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right] T_2}} + \sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\, T_1 \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}} \cdots\right.}{\sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\,(-1+T_1 T_2)} \\ \Sigma & 1 \end{pmatrix}$$

**t2 = (V // Γ) /. T_a_ :> 1 / T_a**

$$\begin{pmatrix} \frac{\left(\frac{-1+\frac{1}{T_1}}{\mathrm{Log}\left[\frac{1}{T_1}\right]}\right)^{1/4}\left(\frac{-1+\frac{1}{T_2}}{\mathrm{Log}\left[\frac{1}{T_2}\right]}\right)^{1/4}}{\left(\frac{-1+\frac{1}{T_1 T_2}}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right]}\right)^{1/4}} & s_1 \\ s_1 & -\frac{\mathrm{Log}\left[\frac{1}{T_1}\right]\left(\mathrm{Log}\left[\frac{1}{T_2}\right]\sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\, T_1 \sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right] T_2}} + \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}} - T_1 \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}}\right)}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right](-1+T_1)\sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}}} \quad -\frac{\mathrm{Log}\left[\frac{1}{T_1}\right]\left(\sqrt{\cdots}\right.}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right.\cdots} \\ s_2 & \frac{\mathrm{Log}\left[\frac{1}{T_2}\right]\left(-\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right] T_2}} + \sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\, T_1 \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}}\right)}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right]\sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\, T_1 \sqrt{\frac{1-T_1 T_2}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right] T_1 T_2}}} \quad \frac{\mathrm{Log}\left[\frac{1}{T_1}\right]\sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right] T_1^2}}\cdots}{\mathrm{Log}\left[\frac{1}{T_1 T_2}\right]\cdots} \\ \Sigma & 1 \end{pmatrix}$$

**t1 == t2 // Simplify**

True

```
t2^-1 ** (V // Γ) // ΓCollect[FullSimplify]
```

$$\left(\begin{array}{cc} \multicolumn{2}{l}{\dfrac{\left(\frac{-1+T_1}{\mathrm{Log}[T_1]}\right)^{1/4}\left(\frac{-1+\frac{1}{T_1T_2}}{\mathrm{Log}\left[\frac{1}{T_1T_2}\right]}\right)^{1/4}\left(\frac{-1+T_2}{\mathrm{Log}[T_2]}\right)^{1/4}}{\left(\frac{-1+\frac{1}{T_1}}{\mathrm{Log}\left[\frac{1}{T_1}\right]}\right)^{1/4}\left(\frac{-1+\frac{1}{T_2}}{\mathrm{Log}\left[\frac{1}{T_2}\right]}\right)^{1/4}\left(\frac{-1+T_1T_2}{\mathrm{Log}[T_1T_2]}\right)^{1/4}}} \\ s_1 & \dfrac{\mathrm{Log}\left[\frac{1}{T_1T_2}\right]\left(\mathrm{Log}[T_2]\,T_1\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right]T_2}}-\mathrm{Log}[T_2]\,T_1^2\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right]T_2}}+\ldots\right.}{\left(\mathrm{Log}\left[\frac{1}{T_1}\right]+\mathrm{Log}\left[\frac{1}{T_2}\right]\right)\mathrm{Log}[T\ldots} \\ s_2 & \dfrac{\mathrm{Log}\left[\frac{1}{T_1T_2}\right]\left(T_1\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right]T_2}}-T_1^2\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right]T_2}}-T_1\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right]T_2}}\,T_2+T_1^2\sqrt{\frac{1-T_2}{\mathrm{Log}\left[\frac{1}{T_2}\right]T_2}}\,T_2+\sqrt{\frac{1-T_1}{\mathrm{Log}\left[\frac{1}{T_1}\right]T_1^2}}\,T\ldots\right.}{\left(\mathrm{Log}\left[\frac{1}{T_1}\right]+\mathrm{Log}\left[\frac{1}{T_2}\right]\right)\mathrm{Log}[T_1T_2]} \\ \Sigma & \end{array}\right.$$

```
t3 = V // A // dA[All] // dS[All]
```

A very large output was generated. Here is a sample of it:

$$\left(\begin{array}{c} \dfrac{32\times 2^{1/4}\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]^5\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]^5 c_1^5\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{3/4}-16\times 2^{1/4}\,\mathrm{Sinh}\left[\frac{\ll 1\gg}{2}\right]^4\mathrm{Sinh}[\ll 1\gg\ \ll 1\gg]^4\,\mathrm{Sinh}\left[\frac{\ll 1\gg}{2}\right]c_1^3\left(\frac{\mathrm{Sinh}\left[\frac{\ll 1\gg}{2}\right]}{c_2}\right)^{3/4}c_2+\ldots}{\ll 239\gg+2\sqrt{2}\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]\mathrm{Sinh}\left[\frac{c_2}{2}\right]^4\left(\frac{\mathrm{Sinh}[\ldots}{c_1}\ldots\right.} \\ t[1] \\ t[2] \end{array}\right.$$

Show Less | Show More | Show Full Output | Set Size Limit...

```
t4 = (V // A) /. c_a_ :→ -c_a
```

$$\left(\begin{array}{cc} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{-c_1-c_2}\right)^{1/4}} & h[1] \\ t[1] & \frac{\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2-2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{c_1+c_2}}} \\ t[2] & \frac{-\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]}{c_1+c_2}}} \quad \frac{c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-e^{c_1+c_2}\,c_1\sqrt{\;\cdots}}{\cdots} \end{array}\right)$$

```
(t4 // dA[All]) ** (V // A)  // ACollect
```

A very large output was generated. Here is a sample of it:

$$\mathrm{ACollect}\left[\left(\begin{array}{c} \frac{-4\,e^{\frac{c_1}{2}}\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\mathrm{Sinh}\left[\frac{1}{2}(-c_1-c_2)\right]\mathrm{Sinh}\left[\frac{c_2}{2}\right]c_1+\ll 17\gg+4\sqrt{2}\,e^{\frac{3c_1}{2}+c_2}\,\mathrm{Sinh}\left[\frac{c_2}{2}\right]\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2^2\sqrt{-\frac{\mathrm{Sinh}\left[\frac{1}{2}\cdots\right.}{c_1\cdots}}\cdots}{\ll 23\gg+2\,e^{\frac{3c_1}{2}+c_2}\,\mathrm{Sinh}\left[\frac{c_2}{2}\right]c_1\,c_2^2\left(-\frac{\mathrm{Sinh}\left[\frac{1}{2}(\ll 1\gg)\right]}{c_1+c_2}\right)^{3/4}\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} \\ t[1] \\ t[2] \end{array}\right)\right.$$

Show Less | Show More | Show Full Output | Set Size Limit...

```
t3 == t4 // Simplify
```

```
$Aborted
```