A NOTE ON THE UNITARITY PROPERTY OF THE GASSNER INVARIANT

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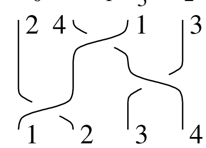
Abstract. We give a 3-page description of the Gassner invariant / representation of braids / pure braids, along with a description and proof of its unitarity property.

The unitarity of the Gassner representation [Ga] of the pure braid group was discussed by many authors (e.g. [Lo, Ab, KLW]) and from several points of view, yet without exposing how utterly simple the formulas turn out to be[1]. When the present author needed quick and easy formulas, he couldn't find them. This note is written in order to rectify this situation. I was heavily influenced by a similar discussion of the unitarity of the Burau representation in [KT, Section 3.1.2].

Let n be a natural number. The braid group B_n on n strands is the group with generators σ_i, for $1 \le i \le n-1$, and with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \le i \le n-2$. The standard way to depict braids, namely elements of B_n, appears on the right. Braids are made of strands that are indexed 1 through n at the bottom. The generator σ_i denotes a positive crossing between the strand at position #i as counted just below the horizontal level of that crossing, and the strand just to its right. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.

$b_0 = \sigma_1\sigma_3^{-1}\sigma_2$:



Let t be a formal variable and let $U_i(t) = U_{n,i}(t)$ denote the $n\times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$. Let $U_i^{-1}(t)$ be the inverse of $U_i(t)$; it is the $n\times n$ identity matrix with the block at $\{i, i+1\}\times\{i, i+1\}$ replaced by $\begin{pmatrix} 0 & \bar t \\ 1 & 1-\bar t \end{pmatrix}$, where $\bar t$ denotes t^{-1}.

$$U_{5,3}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

~~Given~~ Let b be a braid $b = \prod_{\alpha=1}^k \sigma_{i_\alpha}^{s_\alpha}$, where the s_α are signs and where products are taken from left to right. Let j_α be the index of the "over" strand at crossing #α in b. The Gassner invariant $\Gamma(b)$ of b is given by the formula on the right ?

$$\Gamma(b) := \prod_{\alpha=1}^{k} U_{i_\alpha}^{s_\alpha}(t_{j_\alpha}).$$

It is a Laurent polynomial in n formal variables $t_1, \ldots, t_n$, with coefficients in $\mathbb{Z}$.

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[1]Partially this is because the formulas are simplest when extended a "Gassner invariant" defined on the full braid group, but then it is not a representation and it is not unitary. Yet it has an easy "unitarity property"; see below.

For example, $\Gamma(\sigma_1\sigma_2\sigma_1) = U_1(t_1)U_2(t_1)U_1(t_2)$ while $\Gamma(\sigma_2\sigma_1\sigma_2) = U_2(t_2)U_1(t_1)U_2(t_1)$. The equality of these two matrix products constitutes the bulk of the proof of the well-definedness of Γ, and the rest is even easier. The verification of this equality is a routine exercise. Impatient readers may find it in the Mathematica notebook that accompanies this note, [BN].

in 3x3 matrix multiplication

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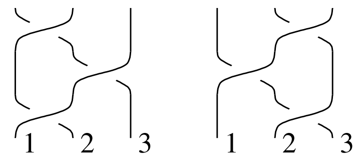
$\sigma_1\sigma_2\sigma_1$: $\sigma_2\sigma_1\sigma_2$:



A second example is the braid b_0 of the first figure. Here and in [BN],

$$\Gamma(b_0) = U_1(t_1)U_3^{-1}(t_4)U_2(t_1) = \begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{t}_4 \\ 0 & t_1 & 0 & 1-\bar{t}_4 \end{pmatrix}$$

Given a permutation $\tau = [\tau 1, \ldots, \tau n]$ of $1, \ldots, n$, let Ω_τ be the triangular $n \times n$ matrix shown on the right ($\frac{1}{1-t_{\tau i}}$ on the diagonal, 1's below the diagonal, 0's above). Let ι denote the identity permutation $[1, 2, \ldots, n]$.

$$\Omega_\tau := \begin{pmatrix} \frac{1}{1-t_{\tau 1}} & 0 & \cdots & 0 \\ 1 & \frac{1}{1-t_{\tau 2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1}{1-t_{\tau n}} \end{pmatrix}$$

Theorem. Let b be a braid that induces a strand permutation $\tau = [\tau 1, \ldots, \tau n]$ (meaning, the strand indices that appear at the top of b are $\tau 1, \tau 2, \ldots, \tau n$). Let $\gamma = \Gamma(b)$ be the Gassner invariant of b. Then γ satisfies the "unitarity property"

$$(1) \qquad \Omega_\tau \gamma^{-1} = \bar{\gamma}^T \Omega_\iota, \qquad \text{or equivalently,} \qquad \gamma^{-1} = \Omega_\tau^{-1}\bar{\gamma}^T\Omega_\iota,$$

where $\bar{\gamma}$ is γ subject to the substitution $\forall i\; t_i \to \bar{t}_i := t_i^{-1}$, and $\bar{\gamma}^T$ is the transpose matrix of $\bar{\gamma}$.

Proof. A direct and simple-minded computation for $b = \sigma_i$ and for $b = \sigma_i^{-1}$, namely for $\gamma = U_i(t_i)$ and for $\gamma = U_i^{-1}(t_{i+1})$ (impatient readers see [BN]), and then, clearly, using the second form of Equation (1), the statement generalizes to products with all the intermediate $\Omega_\tau^{-1}\Omega_\tau$ pairs cancelling out nicely. □

Proves Eqn (1)

If the Gassner invariant Γ is restricted to pure braids, namely to braids that induces the identity permutation, it becomes multiplicative and then it can be called "the Gassner representation" (in general Γ can be recast as a homomorphism into $M_{n\times n}(\mathbb{Z}[t_i, \bar{t}_i]) \rtimes S_n$, where S_n acts on matrices by permuting the variables t_i appearing in their entries).

For pure braids $\Omega_\tau = \Omega_\iota$ and hence by conjugating (in the $t_i \to 1/t_i$ sense) and transposing Equation (1) and replacing γ by γ^{-1}, we find that the theorem also holds if Ω is replaced with $\bar{\Omega}^T$, and hence also with $\Omega + \bar{\Omega}^T$, which is formally Hermitian. Extending the coefficients to $\mathbb{C}$, we find that the same is true for $\Psi := i\Omega - i\bar{\Omega}^T$.

If the t_i's are specialized to complex numbers of unit norm then inversion is the same as complex conjugation. If also the t_i's are sufficiently close to 1, then $\Omega + \bar{\Omega}^T$ is dominated by its main diagonal which is real and large, and hence it is positively definite and genuinely Hermitian. Thus in that case, the Gassner representation is unitary in the standard sense of the word, relative to the inner product on $\mathbb{C}^n$ defined by $\cancel{\Omega + \bar{\Omega}^T}$ Ψ.

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positive, and have positive imaginary parts

Ψ Ψ Ψ

References

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[Ga] B. J. Gassner, *On Braid Groups,* Ph.D. thesis, New York Univeristy, 1959.

[KT] C. Kassel and V. Turaev, *Braid Groups,* Springer GTM **247**, 2008.

$i(iy) - i(\overline{iy}) = -y - y = -2y$

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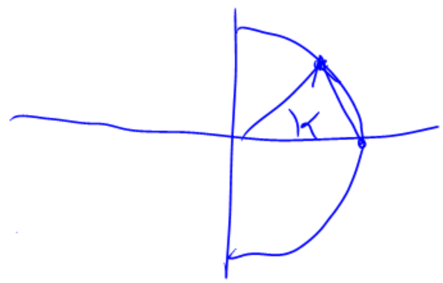
$$\mathrm{Re}\left(\frac{1}{1-t}\right) = \frac{\mathrm{Re}(1-\bar{t})}{\|1-t\|^2}$$



$$= \frac{1-\cos\alpha}{(\cos\alpha-1)^2+\sin^2\alpha}$$

$$= \frac{1-\cos\alpha}{2-2\cos\alpha} = \frac{1}{2}$$

A. Our closing remark is that the Gassner representation easily extends to pure w-braids (e.g. [..]), where the generators σ_{ij} are defined, by setting $\sigma_{ij}^{\pm1} = U_{ij}^{\pm1}(t_i)$ and where $U_{ij}(t)$ is .... . Yet on w-braids the Gassner representation does not satisfy the unitarity property of this note, and I'd be very surprised if it is at all unitary.

Pensieve header: Mathemaica notebook accompanying “A Note on the Unitarity Property of the Gassner Invariant” by Dror Bar-Natan, http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner/..

Definitions.

Try to clean

```
U_i_[t_] := ReplacePart[
   IdentityMatrix[n],
   {{i, i} → 1 - t, {i, i + 1} → 1,
    {i + 1, i} → t, {i + 1, i + 1} → 0}
  ];
Uinv_i_[t_] := Inverse[U_i[t]];
Ω_τ_ := Table[
   Which[i < j, 0, i == j, 1/(1 - t_τ[[i]]), i > j, 1],
   {i, n}, {j, n}];
OverBar[x_] := x /. t_i_ :> 1 / t_i;
```

U_i and U_i^{-1}. The named matrices.

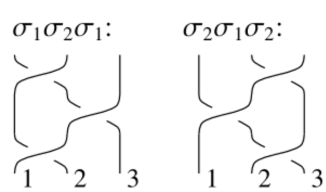
```
n = 5; MatrixForm /@ Simplify /@ {U_2[t], Uinv_2[t]}
```

3 3

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1-t & 1 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 \\ 0 & 1 & \frac{-1+t}{t} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {Ω_{2,3,1}, Inverse[Ω_{2,3,1}]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-t_2} & 0 & 0 \\ 1 & \frac{1}{1-t_3} & 0 \\ 1 & 1 & \frac{1}{1-t_1} \end{pmatrix}, \begin{pmatrix} 1-t_2 & 0 & 0 \\ -(-1+t_2)(-1+t_3) & 1-t_3 & 0 \\ -(-1+t_1)(-1+t_2)\,t_3 & -(-1+t_1)(-1+t_3) & 1-t_1 \end{pmatrix}\right\}$$

The R3 move  .


```
n = 3; MatrixForm /@ Simplify /@ {U_1[t_1].U_2[t_1].U_1[t_2], U_2[t_2].U_1[t_1].U_2[t_1]}
```

$$\left\{\begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}\right\}$$

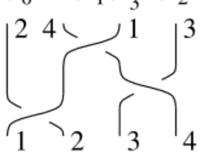
The unitarity property for the generators.

$\texttt{n = 5; } \gamma \texttt{ = } U_3[t_3]\texttt{;}$
$\texttt{MatrixForm /@ Simplify /@ } \{\Omega_{\{1,2,4,3,5\}}.\texttt{Inverse}[\gamma],\ \texttt{Transpose}[\overline{\gamma}].\Omega_{\{1,2,3,4,5\}}\}$

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3\,t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3\,t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

$\texttt{n = 5; } \gamma \texttt{ = } \texttt{Uinv}_3[t_4]\texttt{;}$
$\texttt{MatrixForm /@ FullSimplify /@ } \{\Omega_{\{1,2,4,3,5\}}.\texttt{Inverse}[\gamma],\ \texttt{Transpose}[\overline{\gamma}].\Omega_{\{1,2,3,4,5\}}\}$

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3\,t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3\,t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

The braid $b_0 = \sigma_1\sigma_3^{-1}\sigma_2$:  .

$\texttt{n = 4; MatrixForm}[\gamma_0 = U_1[t_1].\texttt{Uinv}_3[t_4].U_2[t_1]]$

$$\begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_4} \\ 0 & t_1 & 0 & -\frac{1-t_4}{t_4} \end{pmatrix}$$

The unitarity property for b_0.

$\texttt{MatrixForm /@ Simplify /@ } \{\Omega_{\{2,4,1,3\}}.\texttt{Inverse}[\gamma_0],\ \texttt{Transpose}[\overline{\gamma_0}].\Omega_{\{1,2,3,4\}}\}$

$$\left\{\begin{pmatrix} 0 & \frac{1}{t_1-t_1\,t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1\,t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3\,(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{t_1-t_1\,t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1\,t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3\,(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}\right\}$$