Pensieve header: Mathemaica notebook accompanying “A Note on the Unitarity Property of the Gassner Invariant” by Dror Bar-Natan, http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner/.

Definitions.

```
U_i_[t_] := ReplacePart[IdentityMatrix[n],
   {{i, i} → 1 - t, {i, i + 1} → 1,
    {i + 1, i} → t, {i + 1, i + 1} → 0}];
U̅_i_[t_] := Inverse[U_i[t]];
Ω[τ___] := Table[
   Which[i < j, 0, i == j, (1 - t_{τ}〚i〛)^-1, i > j, 1],
   {i, n}, {j, n}];
X̅_ := X /. t_i_ :→ 1 / t_i;
U_i_,j_ := ReplacePart[IdentityMatrix[n],
   {{i, i} → 1, {i, j} → 1 - t_i,
    {j, i} → 0, {j, j} → t_i}];
V_i_,j_ := ReplacePart[IdentityMatrix[n],
   {{i, i} → 1, {i, j} → 1 - t_j,
    {j, i} → 0, {j, j} → t_i}];
DD := DiagonalMatrix[Table[1 - t_i, {i, n}]];
```

The named matrices.

```
n = 5; MatrixForm /@ Simplify /@ {U_3[t], U̅_3[t]}
```

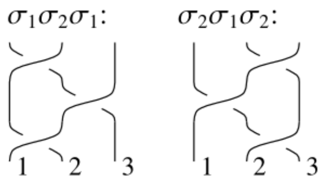
$$\left\{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1-t&1&0\\0&0&t&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&\frac{1}{t}&0\\0&0&1&\frac{-1+t}{t}&0\\0&0&0&0&1\end{pmatrix}\right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {Ω[2, 3, 1], Inverse[Ω[2, 3, 1]]}
```

$$\left\{\begin{pmatrix}\frac{1}{1-t_2}&0&0\\1&\frac{1}{1-t_3}&0\\1&1&\frac{1}{1-t_1}\end{pmatrix},\begin{pmatrix}1-t_2&0&0\\-(-1+t_2)(-1+t_3)&1-t_3&0\\-(-1+t_1)(-1+t_2)t_3&-(-1+t_1)(-1+t_3)&1-t_1\end{pmatrix}\right\}$$

```
n = 5; MatrixForm /@ {U_4,1, V_4,1, DD}
```

$$\left\{\begin{pmatrix}t_4&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1-t_4&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}t_4&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1-t_1&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1-t_1&0&0&0&0\\0&1-t_2&0&0&0\\0&0&1-t_3&0&0\\0&0&0&1-t_4&0\\0&0&0&0&1-t_5\end{pmatrix}\right\}$$

The R3 move 

.

```
n = 3; MatrixForm /@ Simplify /@ {U1[t1].U2[t1].U1[t2], U2[t2].U1[t1].U2[t1]}
```

$$\left\{\begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}\right\}$$

The unitarity property for the generators.

```
n = 5; γ = U3[t3];
MatrixForm /@
 Simplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

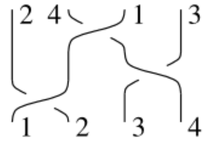
$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

```
n = 5; γ = Ū3[t4];
MatrixForm /@
 FullSimplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

$b_0 = \sigma_1 \sigma_3^{-1} \sigma_2$:

The braid



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```
n = 4; MatrixForm[γ_0 = U_1[t_1].U_3^-[t_4].U_2[t_1]]
```

$$\begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_4} \\ 0 & t_1 & 0 & -\frac{1-t_4}{t_4} \end{pmatrix}$$

The unitarity property for b_0.

```
MatrixForm /@ Simplify /@ {Ω[2, 4, 1, 3].Inverse[γ_0], Transpose[overline(γ_0)].Ω[1, 2, 3, 4]}
```

$$\left\{ \begin{pmatrix} 0 & \frac{1}{t_1 - t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1 - t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{t_1 - t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1 - t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix} \right\}$$

On to w-braids

```
n = 3; MatrixForm /@ Simplify /@ {U_{1,2}.U_{1,3}.U_{2,3}, U_{2,3}.U_{1,3}.U_{1,2}}
```

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix} \right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {U_{1,2}.U_{1,3}, U_{1,3}.U_{1,2}}
```

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix} \right\}$$

The “Other Gassner” Γ’

```
n = 4; MatrixForm /@ Simplify /@ {V_{4,1}, Inverse[DD].U_{4,1}.DD}
```

$$\left\{ \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix} \right\}$$

```
n = 4; MatrixForm /@ Simplify /@ {
  Transpose[DD̄].Ω[1, 2, 3, 4].DD,
  Ψ' = ⅈ Transpose[DD̄].(Ω[1, 2, 3, 4] - Transpose[Ω[1, 2, 3, 4]̄]).DD
 }
```

$$\left\{\begin{pmatrix} 1-\frac{1}{t_1} & 0 & 0 & 0 \\ (1-t_1)\left(1-\frac{1}{t_2}\right) & 1-\frac{1}{t_2} & 0 & 0 \\ (1-t_1)\left(1-\frac{1}{t_3}\right) & (1-t_2)\left(1-\frac{1}{t_3}\right) & 1-\frac{1}{t_3} & 0 \\ (1-t_1)\left(1-\frac{1}{t_4}\right) & (1-t_2)\left(1-\frac{1}{t_4}\right) & (1-t_3)\left(1-\frac{1}{t_4}\right) & 1-\frac{1}{t_4} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{i\left(-1+t_1^2\right)}{t_1} & i\left(-1+\frac{1}{t_1}\right)(1-t_2) & i\left(-1+\frac{1}{t_1}\right)(1-t_3) & i\left(-1+\frac{1}{t_1}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_2)}{t_2} & \frac{i\left(-1+t_2^2\right)}{t_2} & i\left(-1+\frac{1}{t_2}\right)(1-t_3) & i\left(-1+\frac{1}{t_2}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_3)}{t_3} & -\frac{i(-1+t_2)(-1+t_3)}{t_3} & \frac{i\left(-1+t_3^2\right)}{t_3} & i\left(-1+\frac{1}{t_3}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_4)}{t_4} & -\frac{i(-1+t_2)(-1+t_4)}{t_4} & -\frac{i(-1+t_3)(-1+t_4)}{t_4} & \frac{i\left(-1+t_4^2\right)}{t_4} \end{pmatrix}\right\}$$