A NOTE ON THE UNITARITY PROPERTY OF THE GASSNER INVARIANT

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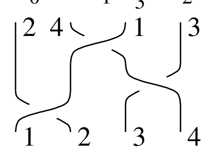
Abstract. We give a 3-page description of the Gassner invariant (or representation) of braids (or pure braids), along with a description and a proof of its unitarity property.

The unitarity of the Gassner representation [Ga] of the pure braid group was discussed by many authors (e.g. [Lo, Ab, KLW]) and from several points of view, yet without exposing how utterly simple the formulas turn out to be[1]. When the present author needed quick and easy formulas, he couldn't find them. This note is written in order to rectify this situation (but with no discussion of theory). I was heavily influenced by a similar discussion of the unitarity of the Burau representation in [KT, Section 3.1.2].

Let n be a natural number. The braid group B_n on n strands is the group with generators σ_i, for $1 \leq i \leq n-1$, and with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \leq i \leq n-2$. A standard way to depict braids, namely elements of B_n, appears on the right. Braids are made of strands that are indexed 1 through n at the bottom. The generator σ_i denotes a positive crossing between the strand at position $\#i$ as counted just below the horizontal level of that crossing, and the strand just to its right. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.

$b_0 = \sigma_1\sigma_3^{-1}\sigma_2$:



Let t be a formal variable and let $U_i(t) = U_{n;i}(t)$ denote the $n\times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$. Let $U_i^{-1}(t)$ be the inverse of $U_i(t)$; it is the $n \times n$ identity matrix with the block at $\{i, i+1\} \times \{i, i+1\}$ replaced by $\begin{pmatrix} 0 & \bar{t} \\ 1 & 1-\bar{t} \end{pmatrix}$, where $\bar{t}$ denotes t^{-1}.

$$U_{5;3}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let b be a braid $b = \prod_{\alpha=1}^{k} \sigma_{i_\alpha}^{s_\alpha}$, where the s_α are signs and where products are taken from left to right. Let j_α be the index of the "over" strand at crossing $\#\alpha$ in b. The Gassner invariant $\Gamma(b)$ of b is given by the formula on the right. It is a Laurent polynomial in n formal variables $t_1, \ldots, t_n$, with coefficients in $\mathbb{Z}$.

$$\Gamma(b) := \prod_{\alpha=1}^{k} U_{i_\alpha}^{s_\alpha}(t_{j_\alpha})$$

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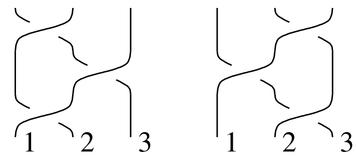
[1]Partially this is because the formulas are simplest when extended a "Gassner invariant" defined on the full braid group, but then it is not a representation and it is not unitary. Yet it has an easy "unitarity property"; see below.

For example, $\Gamma(\sigma_1\sigma_2\sigma_1) = U_1(t_1)U_2(t_1)U_1(t_2)$ while $\Gamma(\sigma_2\sigma_1\sigma_2) = U_2(t_2)U_1(t_1)U_2(t_1)$. The equality of these two matrix products constitutes the bulk of the proof of the well-definedness of Γ, and the rest is even easier. The verification of this equality is a routine exercise in 3×3 matrix multiplication. Impatient readers may find it in the *Mathematica* notebook that accompanies this note, [BN].

$\sigma_1\sigma_2\sigma_1$: $\sigma_2\sigma_1\sigma_2$:



A second example is the braid b_0 of the first figure. Here and in [BN],

$$\Gamma(b_0) = U_1(t_1)U_3^{-1}(t_4)U_2(t_1) = \begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{t}_4 \\ 0 & t_1 & 0 & 1-\bar{t}_4 \end{pmatrix}$$

Given a permutation $\tau = [\tau 1, \dots, \tau n]$ of $1, \dots, n$, let $\Omega(\tau)$ be the triangular $n \times n$ matrix shown on the right (diagonal entries $(1-t_{\tau i})^{-1}$, 1's below the diagonal, 0's above). Let ι denote the identity permutation $[1, 2, \dots, n]$.

$$\Omega(\tau) := \begin{pmatrix} (1-t_{\tau 1})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{\tau 2})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & (1-t_{\tau n})^{-1} \end{pmatrix}$$

Theorem. Let b be a braid that induces a strand permutation $\tau = [\tau 1, \dots, \tau n]$ (meaning, the strand indices that appear at the top of b are $\tau 1, \tau 2, \dots, \tau n$). Let $\gamma = \Gamma(b)$ be the Gassner invariant of b. Then γ satisfies the "unitarity property"

$$(1) \qquad \Omega(\tau)\gamma^{-1} = \bar{\gamma}^T\Omega(\iota), \qquad \text{or equivalently,} \qquad \gamma^{-1} = \Omega(\tau)^{-1}\bar{\gamma}^T\Omega(\iota),$$

where $\bar{\gamma}$ is γ subject to the substitution $\forall i\; t_i \to \bar{t}_i := t_i^{-1}$, and $\bar{\gamma}^T$ is the transpose matrix of $\bar{\gamma}$.

Proof. A direct and simple-minded computation proves Equation (1) for $b = \sigma_i$ and for $b = \sigma_i^{-1}$, namely for $\gamma = U_i(t_i)$ and for $\gamma = U_i^{-1}(t_{i+1})$ (impatient readers see [BN]), and then, clearly, using the second form of Equation (1), the statement generalizes to products with all the intermediate $\Omega(\tau)^{-1}\Omega(\tau)$ pairs cancelling out nicely. □

If the Gassner invariant Γ is restricted to pure braids, namely to braids that induce the identity permutation, it becomes multiplicative and then it can be called "the Gassner representation" (in general Γ can be recast as a homomorphism into $M_{n\times n}(\mathbb{Z}[t_i, \bar{t}_i]) \rtimes S_n$, where S_n acts on matrices by permuting the variables t_i appearing in their entries).

For pure braids $\Omega(\tau) = \Omega(\iota) =: \Omega$ and hence by conjugating (in the $t_i \to 1/t_i$ sense) and transposing Equation (1) and replacing γ by γ^{-1}, we find that the theorem also holds if Ω is replaced with $\bar{\Omega}^T$, and hence also with $\Omega + \bar{\Omega}^T$, which is formally Hermitian. Extending the coefficients to $\mathbb{C}$, we find that the same is true for $\Psi := i\Omega - i\bar{\Omega}^T$.

If the t_i's are specialized to complex numbers of unit norm then inversion is the same as complex conjugation. If also the t_i's are sufficiently close to 1 and have positive imaginary parts, then Ψ is dominated by its main diagonal entries, which are real, positive, and large, and hence Ψ is positive definite and genuinely Hermitian. Thus in that case, the Gassner representation is unitary in the standard sense of the word, relative to the inner product on $\mathbb{C}^n$ defined by Ψ.

Our closing remark is that the Gassner representation easily extends to a representation of pure v/w-braids. See e.g. [BND, Sections 2.1.2 and 2.2], where the generators σ_{ij} are described (they are *not* generators of the ordinary pure braid group). Simply set $\Gamma(\sigma_{ij})^{\pm 1} = U_{ij}^{\pm 1}(t_i)$ where $U_{ij}(t)$ is the $n \times n$ identity matrix with its 2×2 block at rows i and j and columns i and j replaced by

$\begin{pmatrix}1 & 1-t\\ 0 & t\end{pmatrix}$. Yet on v/w-braids Γ does not satisfy the unitarity property of this note and I'd be very surprised if it is at all unitary.

References

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Pensieve header: Mathemaica notebook accompanying “A Note on the Unitarity Property of the Gassner Invariant” by Dror Bar-Natan, http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner/.

Definitions.

```
U_i_[t_] := ReplacePart[
   IdentityMatrix[n],
   {{i, i} → 1 - t, {i, i + 1} → 1,
    {i + 1, i} → t, {i + 1, i + 1} → 0}
  ];
Uinv_i_[t_] := Inverse[U_i[t]];
Ω[τ___] := Table[
   Which[i < j, 0, i == j, (1 - t_{τ}[[i]])^-1, i > j, 1],
   {i, n}, {j, n}];
OverBar[X_] := X /. t_i_ :→ 1 / t_i;
U_i_,j_[t_] := ReplacePart[
   IdentityMatrix[n],
   {{i, i} → 1, {i, j} → 1 - t,
    {j, i} → 0, {j, j} → t}
  ];
```

The named matrices.

```
n = 5; MatrixForm /@ Simplify /@ {U_3[t], Uinv_3[t]}
```

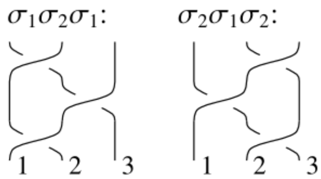
$$\left\{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1-t&1&0\\0&0&t&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&\frac{1}{t}&0\\0&0&1&\frac{-1+t}{t}&0\\0&0&0&0&1\end{pmatrix}\right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {Ω[2, 3, 1], Inverse[Ω[2, 3, 1]]}
```

$$\left\{\begin{pmatrix}\frac{1}{1-t_2}&0&0\\1&\frac{1}{1-t_3}&0\\1&1&\frac{1}{1-t_1}\end{pmatrix},\begin{pmatrix}1-t_2&0&0\\-(-1+t_2)(-1+t_3)&1-t_3&0\\-(-1+t_1)(-1+t_2)t_3&-(-1+t_1)(-1+t_3)&1-t_1\end{pmatrix}\right\}$$

```
n = 5; MatrixForm[U_4,1[t]]
```

$$\begin{pmatrix}t&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1-t&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

The R3 move 

.

```
n = 3; MatrixForm /@ Simplify /@ {U_1[t_1].U_2[t_1].U_1[t_2], U_2[t_2].U_1[t_1].U_2[t_1]}
```

$$\left\{\begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1(-1+t_2) & t_1 & 0 \\ t_1 t_2 & 0 & 0 \end{pmatrix}\right\}$$

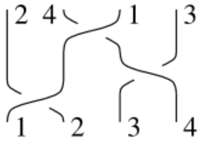
The unitarity property for the generators.

```
n = 5; γ = U_3[t_3];
MatrixForm /@
 Simplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3 t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

```
n = 5; γ = Uinv_3[t_4];
MatrixForm /@
 FullSimplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3 t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

The braid $b_0 = \sigma_1 \sigma_3^{-1} \sigma_2$: 

.

```
n = 4; MatrixForm[γ₀ = U₁[t₁].Uinv₃[t₄].U₂[t₁]]
```

$$\begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_4} \\ 0 & t_1 & 0 & -\frac{1-t_4}{t_4} \end{pmatrix}$$

The unitarity property for b_0.

```
MatrixForm /@ Simplify /@ {Ω[2, 4, 1, 3].Inverse[γ₀], Transpose[γ₀‾].Ω[1, 2, 3, 4]}
```

$$\left\{ \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix} \right\}$$

On to w-braids

```
n = 3; MatrixForm /@ Simplify /@ {U₁,₂[t₁].U₁,₃[t₁].U₂,₃[t₂], U₂,₃[t₂].U₁,₃[t₁].U₁,₂[t₁]}
```

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix} \right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {U₁,₂[t₁].U₁,₃[t₁], U₁,₃[t₁].U₁,₂[t₁]}
```

$$\left\{ \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix} \right\}$$