Oberwolfach Handout on May 19, 2014

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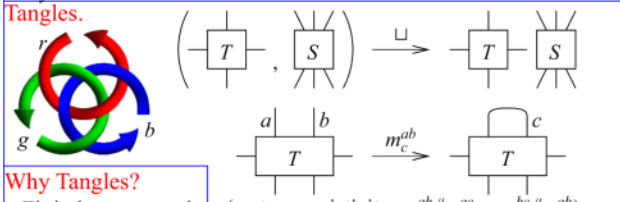
Some very good formulas for the Alexander polynomial, 1

Abstract. I will describe some very good formulas for a *(matrix plus scalar)*-valued extension of the Alexander polynomial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to *(free Lie algebras plus cyclic words)*, and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

Tangles.



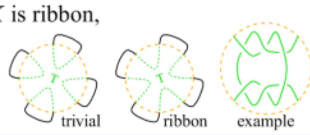
(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon,

$Z(K) \in \{cl_2(Z)\colon cl_1(Z) = 1\}$.

(Genus and crossing number are also definable properties).



Theorem 1. $\exists!$ an invariant γ: {pure framed S-component tangles} $\to R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$1.\ \left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$2.\ \begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\theta\psi/\mu\end{array}\right)_{T_a,T_b\to T_c},$$

and satisfying $(|_a;\ {}_a\!\nearrow\!\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\!\nearrow_a) \xrightarrow{\gamma} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right).$

In Addition, • This is really "just" a stitching formula for Burau/Gassner [LD, KLW, CT].

- $L \mapsto (\omega, A) \mapsto \omega\det'(A-I)/(1-T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally derfined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea: $(\omega, A=(\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab}t_ah_b)$

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Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ε, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ε; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t|h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ/μ  ε + δ θ/μ; ϕ + α ψ/μ  Ξ + θ ψ/μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1 1-T_a; 0 T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1/T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Factor[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_{a_} → a, ∞];
  M = Outer[Factor[∂_{t_#1 h_#2}λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

Meta-Associativity

Runs.

$$\gamma = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_S\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right];$$

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(γ // m_{12→1} // m_{13→1}) == (γ // m_{23→2} // m_{12→1})
True
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R3 … divide and conquer!

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{Rm_{51} Rm_{62} Rp_{34} // m_{14→1} // m_{25→2} // m_{36→3},
 Rp_{61} Rm_{24} Rm_{35} // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$

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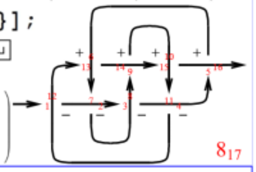
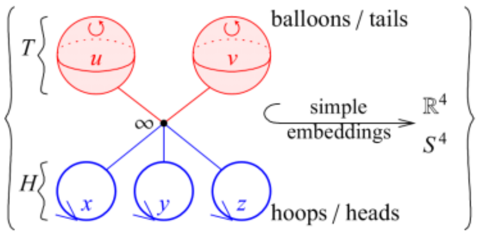
γ = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};
Do[γ = γ // m_{1k→1}, {k, 2, 16}];
γ
```

$$\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} & h_1\\ t_1 & 1\end{pmatrix}$$

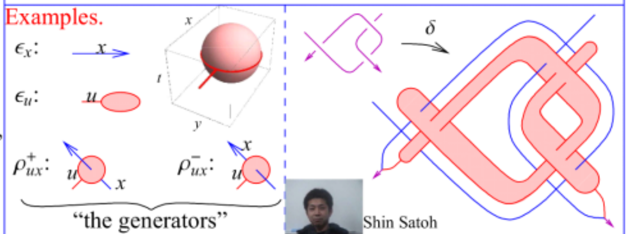


$\mathcal{K}^{bh}(H;T)$.

"Simply-knotted balloons and hoops"

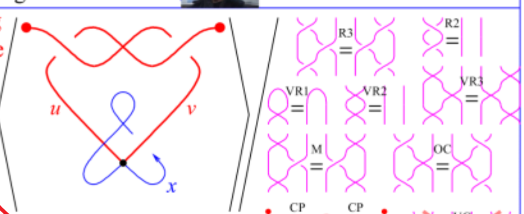


Examples.

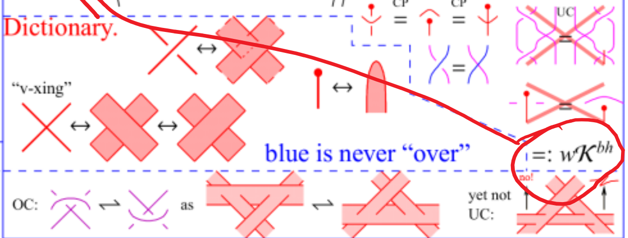


Disturbing Conjecture

$\mathcal{K}^{bh} \stackrel{?}{=}$



Dictionary.



Also add Alex testing to Dugo.

regular Alexander

show 2 o

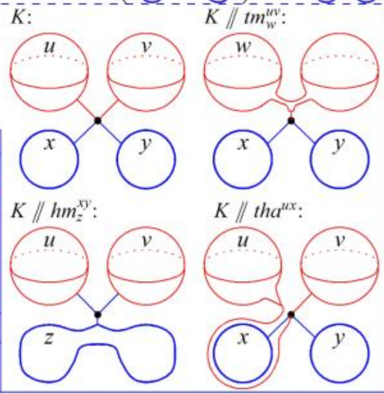
add link

Add: Weaknesses: 1. Operations are non-linear.
2. wA is always a Laurent polynomial, but proving it takes exponentially many conditions.

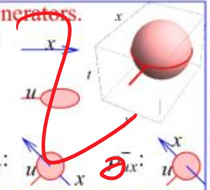
Some very good formulas for the Alexander polynomial, 2

Operations Punctures & Cuts | Connected Sums.

If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.



Generators.



Proposition. The generators generate.

Definition. l_{xu} is the linking number of hoop x with balloon u. For $x \in H$, $\sigma_x := \prod_{u\in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a\in T})$, the ring of rational functions in T variables.

Theorem 2 [BNS]. $\exists!$ an invariant $\beta\colon w\mathcal{K}^{bh}(H;T) \to R \times M_{T\times H}(R)$, intertwining

1. $\left(\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array}$,

2. $\begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \Xi\end{array} \xrightarrow{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \Xi\end{array}\right)_{T_u,T_v\to T_w}$,

3. $\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \Xi\end{array} \xrightarrow{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \Xi\end{array}$,

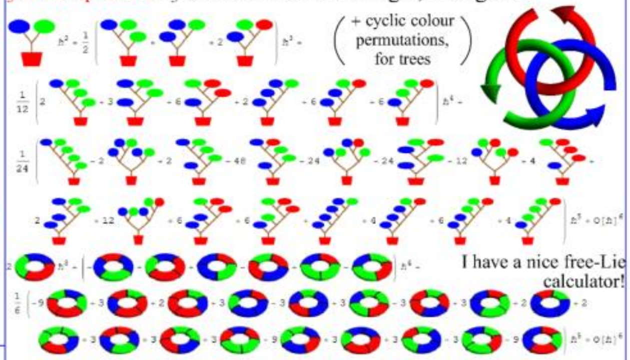
4. $\begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \theta\\ T & \phi & \Xi\end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|cc}\nu\omega & x & H\\ \hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu\\ T & \phi/\nu & \Xi-\phi\theta/\nu\end{array}$,

and satisfying $(\epsilon_x;\epsilon_u;\rho^\pm_{ux}) \xrightarrow{\beta} \left(\begin{array}{c|c}1 & x\\ \hline & \end{array}; \begin{array}{c|c}1 & \\ \hline u & \end{array}; \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}\right)$.

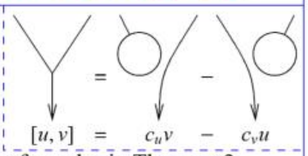
Proposition. If T is a u-tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \operatorname{diag}(\sigma_a)_{a\in S}$.

Theorem 3 [BND, BN]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w-knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.



A Big Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A\rangle.$$

Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

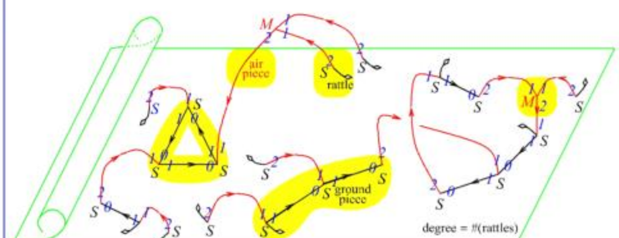
$$O(A,B,\kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^* A}\alpha + \kappa^* B\rangle\right).$$


Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some STU- and IHX-like relations).



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define "finite type" for arbitrary 2-knots.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Add a quick combinatorial ξ?