Pensieve Header: Cheat sheat $\beta$ verification program, continues pensieve://2014-04/, continued pensieve://2014-06/. Seems to be the first occurrence of Γ-calculus.

# Program

```
<< KnotTheory`
```

```
Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
βSimplify = Simplify;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, σ_, μ_]] := B[
   βSimplify[ω], σ,
   Collect[μ, _h, Collect[#, _t, βSimplify] &]
  ];
hL[b_] := Union[Cases[b, h[s_] :> s, Infinity]];
tL[b_] := Union[Cases[b, t[s_] | T_s_ :> s, Infinity]];
dL[b_] := Union[hL[b], tL[b]];
σ_ ⊢ h_ := (∂_h σ /. 0 → 1);
B[ω_, σ_, μ_]@A := Module[
   {tails, heads},
   tails = tL[B[ω, σ, μ]]; heads = hL[B[ω, σ, μ]];
   Outer[βSimplify[∂_{t[#1],h[#2]} μ] &, tails, heads]
  ];
βForm[B[ω_, σ_, μ_]] := Module[
   {tails, heads, mat},
   tails = tL[B[ω, σ, μ]]; heads = hL[B[ω, σ, μ]];
   mat = Outer[βSimplify[∂_{h[#1],t[#2]} μ] &, heads, tails];
   PrependTo[mat, t /@ tails];
   mat = Join[
     {Prepend[h /@ heads, ω]},
     Transpose[mat],
     {Prepend[(σ ⊢ h[#]) & /@ heads, "1+Σ/ω"]}
    ];
   MatrixForm[mat]
  ];
βForm[else_] := else /. b_B :> βForm[b];
Format[b_B, StandardForm] := βForm[b];
B /: B[ω1_, σ1_, μ1_] == B[ω2_, σ2_, μ2_] := Module[
   {heads, tails},
   tails = tL[{B[ω1, σ1, μ1], B[ω2, σ2, μ2]}];
   heads = hL[{B[ω1, σ1, μ1], B[ω2, σ2, μ2]}];
   (ω1 == ω2) && (σ1 == σ2) && (
     And @@ Flatten[Outer[
        (Coefficient[μ1, t[#1] h[#2]] == Coefficient[μ2, t[#1] h[#2]]) &,
        tails, heads
```

```
       ]]
     )
   ];

B /: B[ω1_, σ1_, μ1_] B[ω2_, σ2_, μ2_] := B[ω1 * ω2, σ1 + σ2, ω2 μ1 + ω1 μ2];
tm[x_, y_, z_][b_] := b /. {t[x] → t[z], t[y] → t[z], T_x → T_z, T_y → T_z};
hm[x_, y_, z_][B[ω_, σ_, μ_]] := B[ω,
    h[z] (σ ⊢ h[x]) (σ ⊢ h[y]) + (σ /. h[x] | h[y] → 0),
    h[z] (D[μ, h[x]] + (σ ⊢ h[x]) ∂_h[y] μ) + (μ /. h[x] | h[y] → 0)
   ] // βCollect;
swapth[y_, x_][B[ω_, σ_, μ_]] := Module[
   {α, β, γ, δ},
   (α β
    γ δ) = (Coefficient[μ, t[y] h[x]]  D[μ, t[y]] /. h[x] → 0
            D[μ, h[x]] /. t[y] → 0     μ /. h[x] | t[y] → 0);
   B[ω + α, σ,
     {(σ ⊢ h[x]) t[y], 1}.(α  β
                          γ  ((ω + α) δ - γ * β) / ω).{h[x], 1}] // βCollect
   ];
dm0[x_, y_, z_][b_] := b // swapth[x, y] // hm[x, y, z] // tm[x, y, z];
dm[a_, b_, c_][B[ω0_, σ_, μ_]] := Module[
  {ω, α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, σa, σb},
  ω = ω0 /. {T_a → T_c, T_b → T_c};
  {σa, σb} = {σ ⊢ h[a], σ ⊢ h[b]} /. {T_a → T_c, T_b → T_c};
  (α β θ
   γ δ ϵ
   ϕ ψ Ξ) =
   (∂_{t[a],h[a]} μ                  ∂_{t[a],h[b]} μ                  ∂_t[a] μ /. h[a] | h[b] → 0
    ∂_{t[b],h[a]} μ                  ∂_{t[b],h[b]} μ                  ∂_t[b] μ /. h[a] | h[b] → 0
    ∂_h[a] μ /. t[a] | t[b] → 0     ∂_h[b] μ /. t[a] | t[b] → 0     μ /. t[a] | t[b] | h[a] | h[b] ·
   ) /. {T_a → T_c, T_b → T_c};
  B[ω + β,
    h[c] σa σb + (σ /. h[a] | h[b] → 0 /. {T_a → T_c, T_b → T_c}),
    {t[c], 1}.(γ + σa δ + σb (α + σa β) + (β γ - α δ)/ω    ϵ + σb θ + (β ϵ - δ θ)/ω
               ϕ + σa ψ + (β ϕ - α ψ)/ω                    Ξ + (β Ξ - ψ θ)/ω).{h[c], 1}
    ] // βCollect
 ]

Unprotect[NonCommutativeMultiply];
b1_B ** b2_B := Module[
   {ρ, σ, labels},
   ρ = b1 * (b2 /. {h[s_] :→ h[σ[s]], t[s_] :→ t[σ[s]], T_s_ :→ T_σ[s]});
   labels = dL[{b1, b2}];
   Do[ρ = ρ // dm[s, σ[s], s], {s, labels}];
   ρ
  ];
```

```
βbRp[x_, y_] := B[1, T_x h[y], (T_x - 1) * t[x] h[y]];
βbRm[x_, y_] := B[1, h[y] / T_x, (1 / T_x - 1) * t[x] h[y]];

βbZ[L_] := Module[{s, Z, c, k},
  s = Skeleton[L];
  Z = Times @@ PD[L] /.
    X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], βbRp[l, i], βbRm[j, i]];
  Do[Z = Z // dm[s[[c, 1]], s[[c, k]], s[[c, 1]]], {c, Length[s]},
   {k, 2, Length[s[[c]]]}];
  Z]
```

# R3 for $\beta$-better

```
{βbRp[1, 2], βbRm[1, 2]}
```

$$\left\{\begin{pmatrix} 1 & h[2] \\ t[1] & -1+T_1 \\ 1+\Sigma/\omega & T_1 \end{pmatrix}, \begin{pmatrix} 1 & h[2] \\ t[1] & -1+\frac{1}{T_1} \\ 1+\Sigma/\omega & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
βbRp[1, 2] ** βbRp[1, 3]
```

$$\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ 1+\Sigma/\omega & 1 & T_1 & T_1 \end{pmatrix}$$

```
{βbRp[1, 2] ** βbRp[1, 3] ** βbRp[2, 3], βbRp[2, 3] ** βbRp[1, 3] ** βbRp[1, 2]}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ t[2] & 0 & 0 & T_1(-1+T_2) \\ 1+\Sigma/\omega & 1 & T_1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ t[2] & 0 & 0 & T_1(-1+T_2) \\ 1+\Sigma/\omega & 1 & T_1 & T_1 T_2 \end{pmatrix}\right\}$$

# dm for $\beta$-better

```
{B0 =
  B[ω, {σ_a, σ_b, σ}.{h@a, h@b, h@S}, {t@a, t@b, t@S}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{h@a, h@b, h@S}],
 B0 // dm[a, b, c]}
```

$$\left\{\begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \alpha & \beta & \theta \\ t[b] & \gamma & \delta & \epsilon \\ t[S] & \phi & \psi & \Xi \\ 1+\Sigma/\omega & \sigma_a & \sigma_b & \sigma \end{pmatrix}, \begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \gamma+\frac{\beta\gamma-\alpha\delta}{\omega}+\delta\sigma_a+(\alpha+\beta\sigma_a)\sigma_b & \frac{\beta\epsilon-\delta\theta+\epsilon\omega+\theta\omega\sigma_b}{\omega} \\ t[S] & \frac{\beta\phi-\alpha\psi+\phi\omega+\psi\omega\sigma_a}{\omega} & \frac{\beta\Xi-\theta\psi+\Xi\omega}{\omega} \\ 1+\Sigma/\omega & \sigma_a\sigma_b & \sigma \end{pmatrix}\right\}$$

```
(B0 // swapth[a, b] // hm[a, b, c] // tm[a, b, c]) == (B0 // dm[a, b, c]) // Simplify
True
```

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \gamma+\frac{\beta\gamma-\alpha\delta}{\omega}+\delta\sigma_a+(\alpha+\beta\sigma_a)\sigma_b & \frac{\beta\epsilon-\delta\theta+\epsilon\omega+\theta\omega\sigma_b}{\omega} \\ t[S] & \frac{\beta\phi-\alpha\psi+\phi\omega+\psi\omega\sigma_a}{\omega} & \frac{\beta\Xi-\theta\psi+\Xi\omega}{\omega} \\ \text{"}1+\Sigma/\omega\text{"} & \sigma_a\sigma_b & \sigma \end{pmatrix} \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \gamma+\frac{\beta\gamma-\alpha\delta}{\omega}+\delta\sigma_a+(\alpha+\beta\sigma_a)\sigma_b & \frac{-\delta\theta+\epsilon(\beta+\omega)+\theta\omega}{\omega} \\ t[S] & \frac{-\alpha\psi+\phi(\beta+\omega)+\psi\omega\sigma_a}{\omega} & \Xi+\frac{\beta\Xi-\theta\psi}{\omega} \\ 1+\Sigma/\omega & \sigma_a\sigma_b & \sigma \end{pmatrix}$$

# Back to $\beta$ (and $\beta$-Burau)

$$B0 = B\left[\omega, \{\sigma_a, \sigma_b, \sigma\}.\{h@a, h@b, h@S\}, \{t@a, t@b, t@S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h@a, h@b, h@S\}\right]$$

$$\begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \alpha & \beta & \theta \\ t[b] & \gamma & \delta & \epsilon \\ t[S] & \phi & \psi & \Xi \\ 1+\Sigma/\omega & \sigma_a & \sigma_b & \sigma \end{pmatrix}$$

```
hm[a, b, c][B0]@A // MatrixForm
```

$$\begin{pmatrix} \alpha+\beta\sigma_a & \theta \\ \gamma+\delta\sigma_a & \epsilon \\ \phi+\psi\sigma_a & \Xi \end{pmatrix}$$

## hm and swapth

$$\left(\omega^{-1}\, \text{hm}[a, b, c][B0]@A \text{ /. Thread}[\{\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi\} \to \omega\{\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi\}]\right) \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \alpha+\beta\sigma_a & \theta \\ \gamma+\delta\sigma_a & \epsilon \\ \phi+\psi\sigma_a & \Xi \end{pmatrix}$$

$$\left((\omega+\alpha)^{-1}\, \text{swapth}[a, a][B0]@A \text{ /. Thread}[\{\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi\} \to \omega\{\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi\}]\right) \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \frac{\alpha\sigma_a}{1+\alpha} & \frac{\beta\sigma_a}{1+\alpha} & \frac{\theta\sigma_a}{1+\alpha} \\ \frac{\gamma}{1+\alpha} & -\frac{\beta\gamma}{1+\alpha}+\delta & \epsilon-\frac{\gamma\theta}{1+\alpha} \\ \frac{\phi}{1+\alpha} & -\frac{\beta\phi}{1+\alpha}+\psi & \Xi-\frac{\theta\phi}{1+\alpha} \end{pmatrix}$$

$$(1+\alpha)\left((\omega+\alpha)^{-1}\,\mathtt{swapth[a,a][B0]@A}\ \text{/. Thread}[\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi\}\to \omega\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi\}]\right)\ \text{// FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \alpha\,\sigma_a & \beta\,\sigma_a & \theta\,\sigma_a \\ \gamma & -\beta\,\gamma+\delta+\alpha\,\delta & \epsilon+\alpha\,\epsilon-\gamma\,\theta \\ \phi & -\beta\,\phi+\psi+\alpha\,\psi & \Xi+\alpha\,\Xi-\theta\,\phi \end{pmatrix}$$

## dm

$$\left((\omega+\beta)^{-1}\begin{pmatrix} \gamma+\frac{\beta\gamma-\alpha\delta}{\omega}+\delta\,\sigma_a+(\alpha+\beta\,\sigma_a)\,\sigma_b & \frac{\beta\epsilon-\delta\theta+\epsilon\omega+\theta\omega\sigma_b}{\omega} \\ \frac{\beta\phi-\alpha\psi+\phi\omega+\psi\omega\sigma_a}{\omega} & \frac{\beta\Xi-\theta\psi+\Xi\omega}{\omega} \end{pmatrix}\ \text{/. Thread}[\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi\}\to\omega\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi\}]\right)\ \text{// FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \frac{\gamma+\beta\gamma-\alpha\delta+\alpha\sigma_b+\sigma_a(\delta+\beta\sigma_b)}{1+\beta} & \frac{\epsilon+\beta\epsilon-\delta\theta+\theta\sigma_b}{1+\beta} \\ \frac{\phi+\beta\phi-\alpha\psi+\psi\sigma_a}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\gamma+\beta\gamma-\alpha\delta+\alpha\sigma_b+\sigma_a(\delta+\beta\sigma_b)}{1+\beta} & \frac{\epsilon+\beta\epsilon-\delta\theta+\theta\sigma_b}{1+\beta} \\ \frac{\phi+\beta\phi-\alpha\psi+\psi\sigma_a}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta} \end{pmatrix}-\begin{pmatrix}\gamma & \epsilon \\ \phi & \Xi\end{pmatrix}\ \text{// FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \frac{\alpha(-\delta+\sigma_b)+\sigma_a(\delta+\beta\sigma_b)}{1+\beta} & \frac{\theta(-\delta+\sigma_b)}{1+\beta} \\ \frac{\psi(-\alpha+\sigma_a)}{1+\beta} & -\frac{\theta\psi}{1+\beta} \end{pmatrix}$$

$$\text{Plus}\left[\begin{pmatrix} \frac{\gamma+\beta\gamma-\alpha\delta+\alpha\sigma_b+\sigma_a(\delta+\beta\sigma_b)}{1+\beta} & \frac{\epsilon+\beta\epsilon-\delta\theta+\theta\sigma_b}{1+\beta} \\ \frac{\phi+\beta\phi-\alpha\psi+\psi\sigma_a}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta} \end{pmatrix}\ \text{/. }\{\alpha\to\alpha+\sigma_a,\ \delta\to\delta+\sigma_b,\ \Xi\to\Xi+\sigma\},\ \begin{pmatrix}-\sigma_a\,\sigma_b & 0 \\ 0 & -\sigma\end{pmatrix}\right]\ \text{// FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \gamma-\frac{\alpha\delta}{1+\beta} & \epsilon-\frac{\delta\theta}{1+\beta} \\ \phi-\frac{\alpha\psi}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta} \end{pmatrix}$$

$$(1+\beta)\begin{pmatrix} \gamma-\frac{\alpha\delta}{1+\beta} & \epsilon-\frac{\delta\theta}{1+\beta} \\ \phi-\frac{\alpha\psi}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta} \end{pmatrix}\ \text{// FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \gamma+\beta\,\gamma-\alpha\,\delta & \epsilon+\beta\,\epsilon-\delta\,\theta \\ \phi+\beta\,\phi-\alpha\,\psi & \Xi+\beta\,\Xi-\theta\,\psi \end{pmatrix}$$

$$\text{Plus}\left[\begin{pmatrix} \frac{\gamma+\beta\gamma-\alpha\delta+\alpha\sigma_b+\sigma_a(\delta+\beta\sigma_b)}{1+\beta} & \frac{\epsilon+\beta\epsilon-\delta\theta+\theta\sigma_b}{1+\beta} \\ \frac{\phi+\beta\phi-\alpha\psi+\psi\sigma_a}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta}\end{pmatrix} \text{/. } \{\alpha\to\alpha+\sigma_a-1,\ \delta\to\delta+\sigma_b-1,\ \Xi\to\Xi+\sigma-1\},\ \begin{pmatrix}1-\sigma_a\sigma_b & 0\\ 0 & 1-\sigma\end{pmatrix}\right] \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \frac{\alpha+\beta+\gamma+\beta\gamma+\delta-\alpha\delta}{1+\beta} & \in+\frac{\theta-\delta\theta}{1+\beta} \\ \phi+\frac{\psi-\alpha\psi}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta}\end{pmatrix}$$

$$\text{Plus}\left[\begin{pmatrix} \frac{\gamma+\beta\gamma-\alpha\delta+\alpha\sigma_b+\sigma_a(\delta+\beta\sigma_b)}{1+\beta} & \frac{\epsilon+\beta\epsilon-\delta\theta+\theta\sigma_b}{1+\beta} \\ \frac{\phi+\beta\phi-\alpha\psi+\psi\sigma_a}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta}\end{pmatrix} \text{/. } \{\alpha\to\alpha-1,\ \delta\to\delta-1,\ \Xi\to\Xi-1\},\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right] \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \frac{\alpha+\beta+\gamma+\beta\gamma+\delta-\alpha\delta+(-1+\alpha)\sigma_b+\sigma_a(-1+\delta+\beta\sigma_b)}{1+\beta} & \frac{\in+\beta\in+\theta-\delta\theta+\theta\sigma_b}{1+\beta} \\ \frac{\phi+\beta\phi+\psi-\alpha\psi+\psi\sigma_a}{1+\beta} & \Xi-\frac{\theta\psi}{1+\beta}\end{pmatrix}$$

# Alexander with $\beta$-better

```
{Knot[8, 17] // βbZ, Alexander[Knot[8, 17]][T₁] // βSimplify}
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.

$$\left\{\begin{pmatrix} -8-\frac{1}{T_1^2}+\frac{4}{T_1}+11T_1-8T_1^2+4T_1^3-T_1^4 & h[1] \\ \frac{t[1]}{1+\Sigma/\omega} & \begin{matrix}0\\1\end{matrix}\end{pmatrix},\ 11-\frac{1}{T_1^3}+\frac{4}{T_1^2}-\frac{8}{T_1}-8T_1+4T_1^2-T_1^3\right\}$$

# The MVA with $\beta$-better

```
βbMVA[L_Link] := Module[{Hs, ω, σ, μ, A, M},
  {ω, σ, μ} = List @@ βbZ[L];
  Hs = Rest[h /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h[a_] :> t[a]];
  M = A - ω DiagonalMatrix[((σ ⊢ #) - 1) & /@ Hs];
  Factor[ω^(2-Length@Skeleton@L) Det[M] / (1 - T_(Skeleton[L][[1,1]]))]
]
```

```
Link["L6a4"] // βbZ
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

| $\frac{(T_1 (-1+T_5)(-1+T_9)-T_5(-1+T_9)+T_9)((-1+T_5)(-1+T_9)+T_1(-1+T_5+T_9))}{T_1 T_5 T_9}$ | h[1] | |
|---|---|---|
| t[1] | $\frac{(-1+T_1)(-1+T_5)(T_1(-1+T_5)+T_5(-1+T_9))(-1+T_9)}{T_1 T_5 T_9}$ | $-$ |
| t[5] | $-\frac{(1+T_1(-1+T_5))(-1+T_5)(-1+T_9)}{T_5 T_9}$ | $(-1+T_1$ |
| t[9] | $-\frac{(-1+T_5)(1+T_1(-2+T_9)-T_9)(-1+T_9)}{T_1 T_9}$ | |
| $1+\Sigma/\omega$ | 1 | |

```
βbMVA[Link["L6a4"]]
```

$$-\frac{(-1+T_1)(-1+T_5)(-1+T_9)}{T_1 T_5}$$

```
Factor[(MultivariableAlexander[#][T] /. T[i_] :> T_{Skeleton[#][[i,1]]}) / βbMVA[#]] & /@
 AllLinks[{2, 8}]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\Big\{T_1^2 T_3,\ T_1^{3/2} T_5^{3/2},\ \sqrt{T_1}\, T_5^{3/2},\ T_1^{3/2}\sqrt{T_5},\ T_1^2 T_7^2,\ T_1^2 T_7^2,\ -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2},$$
$$-\frac{\sqrt{T_1}\sqrt{T_5}}{T_9^{3/2}},\ \sqrt{T_1}\sqrt{T_5},\ T_1^{3/2} T_5^{7/2},\ \frac{\sqrt{T_1}}{T_5^{3/2}},\ \frac{\sqrt{T_1}}{T_5^{3/2}},\ T_1 T_7^2,\ \frac{1}{T_7},\ -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}},\ T_1^{3/2} T_5^{7/2},$$
$$\sqrt{T_1}\, T_5^{5/2},\ \sqrt{T_1}\, T_5^{3/2},\ \frac{\sqrt{T_1}}{\sqrt{T_5}},\ T_1^{3/2} T_5^{3/2},\ \sqrt{T_1}\, T_5^{3/2},\ \frac{T_1^{3/2}}{\sqrt{T_5}},\ \frac{T_1^{3/2}}{\sqrt{T_5}},\ T_1^{3/2} T_5^{7/2},\ \frac{T_1}{T_7},\ T_1 T_7,$$
$$T_1^2 T_7^3,\ T_1^2 T_7^3,\ T_1^{5/2} T_9^{5/2},\ T_1^{5/2} T_9^{5/2},\ T_1^{5/2} T_9^{5/2},\ -T_1^{3/2} T_5^{3/2}\sqrt{T_9},\ -\sqrt{T_1},\ -T_1^{3/2} T_5^2 T_{11}^2,$$
$$-\frac{\sqrt{T_1}}{T_{11}^2},\ -\frac{\sqrt{T_1}\, T_5}{T_{11}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{3/2}},\ T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2},\ T_1^{3/2} T_5^{3/2},\ \sqrt{T_1}\sqrt{T_5},\ -T_1^{3/2} T_5^2 T_{11}^2,$$
$$-\sqrt{T_1}\, T_5^2 T_{11},\ -\sqrt{T_1}\, T_5^{3/2} T_{11}^{3/2},\ -T_1^{3/2} T_5^{5/2}\sqrt{T_{13}},\ \frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}\sqrt{T_{13}}},\ \sqrt{T_1}\sqrt{T_5}\sqrt{T_9}\sqrt{T_{13}}\Big\}$$

# Burau Calculus

```
Plus[
  ( β + ω        h[c]                                              h[S]
    t[c]         γ + (β γ - α δ)/ω + δ σa + (α + β σa) σb          (β ϵ - δ θ + ϵ ω + θ ω σb)/ω
    t[S]         (β ϕ - α ψ + ϕ ω + ψ ω σa)/ω                      (β Ξ - θ ψ + Ξ ω)/ω
    "1+Σ/ω"      σa σb                                             σ )  /.
   {α → α + ω σa, δ → δ + ω σb, Ξ → Ξ + ω σ},
  - ( 0  0        0
      0  ω σa σb  0
      0  0        ω σ
      0  0        0 )
 ] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \frac{-\alpha\,\delta+\gamma\,(\beta+\omega)+\beta\,\omega\,\sigma_a\,\sigma_b}{\omega} & \epsilon+\frac{\beta\,\epsilon-\delta\,\theta}{\omega} \\ t[S] & \phi+\frac{\beta\,\phi-\alpha\,\psi}{\omega} & \Xi+\beta\,\sigma+\frac{\beta\,\Xi-\theta\,\psi}{\omega} \\ 1+\Sigma/\omega & \sigma_a\,\sigma_b & \sigma \end{pmatrix}$$

```
Plus[
  ( β + ω        h[c]                                              h[S]
    t[c]         γ + (β γ - α δ)/ω + δ σa + (α + β σa) σb          (β ϵ - δ θ + ϵ ω + θ ω σb)/ω
    t[S]         (β ϕ - α ψ + ϕ ω + ψ ω σa)/ω                      (β Ξ - θ ψ + Ξ ω)/ω
    "1+Σ/ω"      σa σb                                             σ )  /.
   {α → α + ω (σa - 1), δ → δ + ω (σb - 1), Ξ → Ξ + ω (σ - 1)},
  - ( 0  0              0
      0  ω (σa σb - 1)  0
      0  0              ω (σ - 1)
      0  0              0 )
 ] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \alpha+\gamma+\delta+\frac{\beta\,\gamma-\alpha\,\delta}{\omega}+\beta\,\sigma_a\,\sigma_b & \epsilon+\theta+\frac{\beta\,\epsilon-\delta\,\theta}{\omega} \\ t[S] & \phi+\psi+\frac{\beta\,\phi-\alpha\,\psi}{\omega} & \Xi+\beta\,(-1+\sigma)+\frac{\beta\,\Xi-\theta\,\psi}{\omega} \\ 1+\Sigma/\omega & \sigma_a\,\sigma_b & \sigma \end{pmatrix}$$

```
Plus[
  ( β + ω        h[c]                                           h[S]
    t[c]         γ + (β γ - α δ)/ω + δ σa + (α + β σa) σb        (β ϵ - δ θ + ϵ ω + θ ω σb)/ω
    t[S]         (β ϕ - α ψ + ϕ ω + ψ ω σa)/ω                    (β Ξ - θ ψ + Ξ ω)/ω
    "1+Σ/ω"      σa σb                                          σ                           ) /.
   {α → α - ω, δ → δ - ω, Ξ → Ξ - ω},
  + ( 0 0 0
      0 ω 0
      0 0 ω
      0 0 0 )
 ] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \frac{\beta\gamma-\alpha\delta+(\alpha+\gamma+\delta)\,\omega+(\alpha-\omega)\,\omega\,\sigma_b+\omega\,\sigma_a\,(\delta-\omega+\beta\,\sigma_b)}{\omega} & \epsilon+\theta+\frac{\beta\epsilon-\delta\theta}{\omega}+\theta\,\sigma_b \\ t[S] & \phi+\psi+\frac{\beta\phi-\alpha\psi}{\omega}+\psi\,\sigma_a & -\beta+\Xi+\frac{\beta\Xi-\theta\psi}{\omega} \\ 1+\Sigma/\omega & \sigma_a\,\sigma_b & \sigma \end{pmatrix}$$