

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/2014-05/MetaCalculi/"];
<< KnotTheory`
<< MetaCalculi-Program.m

Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.
```

General

```
SXForm[L = Link["L6a4"]]
```

```
KnotTheory:loading: Loading precomputed data in PD4Links`.
```

```
SXForm[{Loop[1, 2, 3, 4], Loop[5, 6, 7, 8], Loop[9, 10, 11, 12]},
  Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]
```

```
Z[L]
```

```
dm[9, 12, 9][
  dm[9, 11, 9][dm[9, 10, 9][dm[5, 8, 5][dm[5, 7, 5][dm[5, 6, 5][dm[1, 4, 1][dm[1, 3, 1][
    dm[1, 2, 1][Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]]]]]]]]]]
```

α -Calculus

```
{Xpab, Xmab} // A
```

$$\left\{ \begin{pmatrix} 1 & h[b] \\ t[a] & \frac{-1+e^{c_a}}{c_a} \end{pmatrix}, \begin{pmatrix} 1 & h[b] \\ t[a] & \frac{e^{-c_a}(1-e^{c_a})}{c_a} \end{pmatrix} \right\}$$

```
{Xm51 Xm62 Xp34 // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
  Xp61 Xm24 Xm35 // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

$$\left\{ \begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix} \right\}$$

$$\alpha = \mathbf{Xm}_{12,1} \mathbf{Xm}_{27} \mathbf{Xm}_{83} \mathbf{Xm}_{4,11} \mathbf{Xp}_{16,5} \mathbf{Xp}_{6,13} \mathbf{Xp}_{14,9} \mathbf{Xp}_{10,15} // \mathbf{A}$$

$$\left(\begin{array}{cccccccccc} 1 & h[1] & h[3] & h[5] & h[7] & h[9] & h[11] & h[13] & h[15] \\ t[2] & 0 & 0 & 0 & \frac{e^{-c_2} (1-e^{c_2})}{c_2} & 0 & 0 & 0 & 0 \\ t[4] & 0 & 0 & 0 & 0 & 0 & \frac{e^{-c_4} (1-e^{c_4})}{c_4} & 0 & 0 \\ t[6] & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_6}}{c_6} & 0 \\ t[8] & 0 & \frac{e^{-c_8} (1-e^{c_8})}{c_8} & 0 & 0 & 0 & 0 & 0 & 0 \\ t[10] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{10}}}{c_{10}} \\ t[12] & \frac{e^{-c_{12}} (1-e^{c_{12}})}{c_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t[14] & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{14}}}{c_{14}} & 0 & 0 & 0 \\ t[16] & 0 & 0 & \frac{-1+e^{c_{16}}}{c_{16}} & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\alpha = \mathbf{Xm}_{12,1} \mathbf{Xm}_{27} \mathbf{Xm}_{83} \mathbf{Xm}_{4,11} \mathbf{Xp}_{16,5} \mathbf{Xp}_{6,13} \mathbf{Xp}_{14,9} \mathbf{Xp}_{10,15} // \mathbf{A};$$

$$\text{Do}[\alpha = \alpha // \text{dm}[1, k, 1], \{k, 2, 16\}]; \alpha$$

$$\left(\begin{array}{c} e^{-3 c_1} (-1 + 4 e^{c_1} - 8 e^{2 c_1} + 11 e^{3 c_1} - 8 e^{4 c_1} + 4 e^{5 c_1} - e^{6 c_1}) \\ t[1] \end{array} \right)$$

Testing R3

$$\{ (\mathbf{Xp}_{12} // \mathbf{A}) ** (\mathbf{Xp}_{13} // \mathbf{A}) ** (\mathbf{Xp}_{23} // \mathbf{A}), (\mathbf{Xp}_{23} // \mathbf{A}) ** (\mathbf{Xp}_{13} // \mathbf{A}) ** (\mathbf{Xp}_{12} // \mathbf{A}) \}$$

$$\left\{ \left(\begin{array}{ccc} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{array} \right), \left(\begin{array}{ccc} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{array} \right) \right\}$$

Testing the KV Solution

The Hard R4 Equation

$$\text{Print} /@ \{ (\mathbf{Xp}_{23} // \mathbf{A}) ** (\mathbf{Xp}_{13} // \mathbf{A}) ** (\mathbf{V} // \mathbf{A}), (\mathbf{V} // \mathbf{A}) ** ((\mathbf{Xp}_{13} // \mathbf{A}) // \text{d}\Delta[1, 1, 2]) \};$$

$$\left(\begin{array}{c}
 \frac{2^{1/4} \left(\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1} \right)^{1/4} \left(\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2} \right)^{1/4}}{\left(\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2} \right)^{1/4}} \\
 \\
 \frac{-\sqrt{2} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} c_2 + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1^2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} \\
 \\
 \frac{\sqrt{2} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} - 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} \\
 \\
 \frac{-e^{\frac{c_1}{2}} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}}}{-e^{\frac{c_1}{2}} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}}}
 \end{array} \right)$$

```

(Xp23 // A) ** (Xp13 // A) ** (V // A) ==
(V // A) ** ((Xp13 // A) // dA[1, 1, 2]) // Simplify
True

```

Unitarity

```

(V // A) ** (Vi // A) // aCollect[Simplify]
(1)

```

qΔ (“renormalized cabling”)

```

qΔ0[z_, x_, y_][a_A] := Module[{b, a},
  Times[
    V // A // dσ[1 → b[x], 2 → b[y]],
    α // dA[z, x, y],
    V // A // dA[1] // dA[2] // dσ[1 → a[x], 2 → a[y]]
  ] // dm[b[x], x, x] // dm[b[y], y, y] // dm[x, a[x], x] // dm[y, a[y], y]
]

```

`Xp13 // A // qΔ[1, 1, 2] // αCollect[Simplify]`

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2} + e^{c_1 + c_2}}{c_1} \\ t[2] & \frac{-1 + e^{c_2}}{c_2} \end{pmatrix}$$

`(Xp23 // A) ** (Xp13 // A)`

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2} + e^{c_1 + c_2}}{c_1} \\ t[2] & \frac{-1 + e^{c_2}}{c_2} \end{pmatrix}$$

`Xp13 // A // qΔ[1, 1, 2] // αCollect[Simplify]`

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2} + e^{c_1 + c_2}}{c_1 + c_2} \\ t[2] & \frac{-1 + e^{c_2}}{c_2} \end{pmatrix}$$

`Clear[ω, α, θ, φ, Ξ];`

`α0 = A[ω, {t[a], t[S]}. $\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}$.{h[a], h[S]}];`

`α1 = α0 // qΔ[a, x, y] // αCollect[Simplify]`

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y} \theta c_x + e^{c_x + c_y} \theta c_x - e^{c_y} \theta c_y + e^{c_x + c_y} \theta c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x - e^{c_y} \alpha c_y + e^{c_x + c_y} \alpha c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x}{-c_x + e^{c_x + c_y} c_x} \\ t[y] & \frac{-\theta c_x + e^{c_y} \theta c_x - \theta c_y + e^{c_y} \theta c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x - \alpha c_y + e^{c_y} \alpha c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x}{-c_y + e^{c_x + c_y} c_y} \end{pmatrix}$$

`αCollect[(Simplify[#] /. cx + cy → ca) &][α1]`

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[a] & 0 & 0 & 0 \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y} \theta c_a + e^{c_x + c_y} \theta c_a}{-c_x + e^{c_x} c_x} & \frac{-e^{c_y} \alpha c_a + e^{c_x + c_y} \alpha c_a}{-c_x + e^{c_x} c_x} & \frac{-e^{c_y} \alpha c_a + e^{c_x + c_y} \alpha c_a}{-c_x + e^{c_x} c_x} \\ t[y] & \frac{-\theta c_a + e^{c_y} \theta c_a}{-c_y + e^{c_x} c_y} & \frac{-\alpha c_a + e^{c_y} \alpha c_a}{-c_y + e^{c_x} c_y} & \frac{-\alpha c_a + e^{c_y} \alpha c_a}{-c_y + e^{c_x} c_y} \end{pmatrix}$$

`Clear[ω, α, θ, φ, Ξ];`

`α0 = A[ω, {t[a], t[S]}. $\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}$.{h[a], h[S]}];`

`α0 // qΔ[a, x, y] // αCollect[Simplify]`

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y} \theta c_x + e^{c_x + c_y} \theta c_x - e^{c_y} \theta c_y + e^{c_x + c_y} \theta c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x - e^{c_y} \alpha c_y + e^{c_x + c_y} \alpha c_y}{-c_x + e^{c_x + c_y} c_x} & \frac{-e^{c_y} \alpha c_x + e^{c_x + c_y} \alpha c_x - e^{c_y} \alpha c_y + e^{c_x + c_y} \alpha c_y}{-c_x + e^{c_x + c_y} c_x} \\ t[y] & \frac{-\theta c_x + e^{c_y} \theta c_x - \theta c_y + e^{c_y} \theta c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x - \alpha c_y + e^{c_y} \alpha c_y}{-c_y + e^{c_x + c_y} c_y} & \frac{-\alpha c_x + e^{c_y} \alpha c_x - \alpha c_y + e^{c_y} \alpha c_y}{-c_y + e^{c_x + c_y} c_y} \end{pmatrix}$$

Γ-Calculus

{Xp_{ab}, Xm_{ab}} // Γ

$$\left\{ \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & 1 - T_a \\ s_b & 0 & T_a \\ \Sigma & 1 & T_a \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & \frac{-1+T_a}{T_a} \\ s_b & 0 & \frac{1}{T_a} \\ \Sigma & 1 & \frac{1}{T_a} \end{pmatrix} \right\}$$

Meta-Associativity

n = 4; γ0 = Γ[ω, ∑_{a=0}ⁿ h_a σ_a, ∑_{a=1}ⁿ ∑_{b=1}ⁿ t_a h_b α_{10 a+b}]

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

γ0 // dm[1, 2, 1] // dm[1, 3, 1]

$$\begin{pmatrix} \omega (1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}) & s_1 & \frac{\alpha_{31} - \alpha_{12} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{11} \alpha_{32} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ s_1 & & \frac{\alpha_{41} - \alpha_{12} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} - \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{11} \alpha_{42} + \alpha_{13} \alpha_{21} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{42} + \alpha_{21} \alpha_{43} - \alpha_{12} \alpha_{21} \alpha_{43} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ s_4 & & \frac{\alpha_{41} - \alpha_{12} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} - \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{11} \alpha_{42} + \alpha_{13} \alpha_{21} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{42} + \alpha_{21} \alpha_{43} - \alpha_{12} \alpha_{21} \alpha_{43} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \Sigma & & \sigma_1 \sigma_2 \sigma_3 \end{pmatrix}$$

γ0 // dm[2, 3, 2] // dm[1, 2, 1]

$$\begin{pmatrix} \omega (1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}) & s_1 & \frac{\alpha_{31} - \alpha_{12} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{23} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{11} \alpha_{32} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{21} \alpha_{33} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ s_1 & & \frac{\alpha_{41} - \alpha_{12} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} - \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{11} \alpha_{42} + \alpha_{13} \alpha_{21} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{42} + \alpha_{21} \alpha_{43} - \alpha_{12} \alpha_{21} \alpha_{43} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ s_4 & & \frac{\alpha_{41} - \alpha_{12} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} - \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{11} \alpha_{42} + \alpha_{13} \alpha_{21} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{42} + \alpha_{21} \alpha_{43} - \alpha_{12} \alpha_{21} \alpha_{43} + \alpha_{12} \alpha_{23}}{1 - \alpha_{12} - \alpha_{13} \alpha_{22} - \alpha_{23} + \alpha_{12} \alpha_{23}} \\ \Sigma & & \sigma_1 \sigma_2 \sigma_3 \end{pmatrix}$$

(γ0 // dm[1, 2, 1] // dm[1, 3, 1]) == (γ0 // dm[2, 3, 2] // dm[1, 2, 1])

True

Column Sums

```
Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];
```

$$\gamma_0 = \Gamma \left[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\} \right];$$

```
γ0 // dm[a, b, c]
```

$$\begin{pmatrix} -(-1 + \beta) \omega & s_c & s_s \\ s_c & \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ s_s & \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \\ \Sigma & \sigma_a \sigma_b & \sigma_s \end{pmatrix}$$

$$\{1, 1\} \cdot \begin{pmatrix} \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \end{pmatrix} // \text{Simplify}$$

$$\left\{ \frac{(-1 + \beta) \gamma + (-1 + \beta) \phi - \alpha (\delta + \psi)}{-1 + \beta}, \frac{\epsilon - \beta \epsilon + \delta \theta + \Xi - \beta \Xi + \theta \psi}{1 - \beta} \right\}$$

$$\{1, 1\} \cdot \begin{pmatrix} \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \end{pmatrix} /. \{\alpha \rightarrow s1 - \gamma - \phi, \delta \rightarrow s2 - \beta - \psi, \Xi \rightarrow s3 - \theta - \epsilon\} // \text{Simplify}$$

$$\left\{ \frac{s1 (-s2 + \beta) + (-1 + s2) (\gamma + \phi)}{-1 + \beta}, \frac{s3 (-1 + \beta) + \theta - s2 \theta}{-1 + \beta} \right\}$$

$$\{1, 1\} \cdot \begin{pmatrix} \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \end{pmatrix} /. \{\alpha \rightarrow s1 - \gamma - \phi, \delta \rightarrow s2 - \beta - \psi, \Xi \rightarrow s3 - \theta - \epsilon\} /.$$

```
s1 | s2 | s3 → 1 // Simplify
```

```
{1, 1}
```

Tangle Concatenation; Γ -inversion

$$n = 3; \{$$

$$\gamma_1 = \Gamma \left[\omega_1, \sum_{a=0}^n h_a \sigma_{1a}, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10 a+b} \right],$$

$$\gamma_2 = \Gamma \left[\omega_2, \sum_{a=0}^n h_a \sigma_{2a}, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \beta_{10 a+b} \right],$$

$$\text{FullStitch}[\gamma_1, \gamma_2], \gamma_1 ** \gamma_2, \text{FullStitch}[\gamma_1, \gamma_2] == \gamma_1 ** \gamma_2 \}$$

$$\left\{ \begin{pmatrix} \omega_1 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Sigma & \sigma_{11} & \sigma_{12} & \sigma_{13} \end{pmatrix}, \begin{pmatrix} \omega_2 & s_1 & s_2 & s_3 \\ s_1 & \beta_{11} & \beta_{12} & \beta_{13} \\ s_2 & \beta_{21} & \beta_{22} & \beta_{23} \\ s_3 & \beta_{31} & \beta_{32} & \beta_{33} \\ \Sigma & \sigma_{21} & \sigma_{22} & \sigma_{23} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \omega_1 \omega_2 & & s_1 & & s_2 & & s_3 \\ s_1 & \alpha_{11} \beta_{11} + \alpha_{21} \beta_{12} + \alpha_{31} \beta_{13} & \alpha_{12} \beta_{11} + \alpha_{22} \beta_{12} + \alpha_{32} \beta_{13} & \alpha_{13} \beta_{11} + \alpha_{23} \beta_{12} + \alpha_{33} \beta_{13} \\ s_2 & \alpha_{11} \beta_{21} + \alpha_{21} \beta_{22} + \alpha_{31} \beta_{23} & \alpha_{12} \beta_{21} + \alpha_{22} \beta_{22} + \alpha_{32} \beta_{23} & \alpha_{13} \beta_{21} + \alpha_{23} \beta_{22} + \alpha_{33} \beta_{23} \\ s_3 & \alpha_{11} \beta_{31} + \alpha_{21} \beta_{32} + \alpha_{31} \beta_{33} & \alpha_{12} \beta_{31} + \alpha_{22} \beta_{32} + \alpha_{32} \beta_{33} & \alpha_{13} \beta_{31} + \alpha_{23} \beta_{32} + \alpha_{33} \beta_{33} \\ \Sigma & \sigma_{11} \sigma_{21} & \sigma_{12} \sigma_{22} & \sigma_{13} \sigma_{23} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \omega_1 \omega_2 & & s_1 & & s_2 & & s_3 \\ s_1 & \alpha_{11} \beta_{11} + \alpha_{21} \beta_{12} + \alpha_{31} \beta_{13} & \alpha_{12} \beta_{11} + \alpha_{22} \beta_{12} + \alpha_{32} \beta_{13} & \alpha_{13} \beta_{11} + \alpha_{23} \beta_{12} + \alpha_{33} \beta_{13} \\ s_2 & \alpha_{11} \beta_{21} + \alpha_{21} \beta_{22} + \alpha_{31} \beta_{23} & \alpha_{12} \beta_{21} + \alpha_{22} \beta_{22} + \alpha_{32} \beta_{23} & \alpha_{13} \beta_{21} + \alpha_{23} \beta_{22} + \alpha_{33} \beta_{23} \\ s_3 & \alpha_{11} \beta_{31} + \alpha_{21} \beta_{32} + \alpha_{31} \beta_{33} & \alpha_{12} \beta_{31} + \alpha_{22} \beta_{32} + \alpha_{32} \beta_{33} & \alpha_{13} \beta_{31} + \alpha_{23} \beta_{32} + \alpha_{33} \beta_{33} \\ \Sigma & \sigma_{11} \sigma_{21} & \sigma_{12} \sigma_{22} & \sigma_{13} \sigma_{23} \end{pmatrix}, \text{True} \}$$

$$\gamma_1^{-1}$$

$$\left(\begin{array}{c|c|c} \frac{1}{\omega_1} & & \\ \hline s_1 & \frac{\alpha_{23} \alpha_{32} - \alpha_{22} \alpha_{33}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} & \frac{\alpha_{13} \alpha_{32} - \alpha_{12} \alpha_{33}}{-\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33}} \\ \hline s_2 & \frac{\alpha_{23} \alpha_{31} - \alpha_{21} \alpha_{33}}{-\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33}} & \frac{\alpha_{13} \alpha_{31} - \alpha_{11} \alpha_{33}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} \\ \hline s_3 & \frac{\alpha_{22} \alpha_{31} - \alpha_{21} \alpha_{32}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} & \frac{\alpha_{12} \alpha_{31} - \alpha_{11} \alpha_{32}}{-\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33}} \\ \hline \Sigma & \frac{1}{\sigma_{11}} & \frac{1}{\sigma_{12}} \end{array} \right)$$

$$\gamma_1 ** \gamma_1^{-1}$$

$$\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & 0 \\ s_2 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 1 \\ \Sigma & 1 & 1 & 1 \end{pmatrix}$$

γ_1 // $ds[1]$ // $ds[2]$ // $ds[3]$

$$\left(\begin{array}{c} \frac{\omega_1 (\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33})}{\sigma_1 \sigma_2 \sigma_3} \\ S_1 \\ S_2 \\ S_3 \\ \Sigma \end{array} \right) \begin{array}{c} S_1 \\ \frac{\alpha_{23} \alpha_{32} - \alpha_{22} \alpha_{33}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} \\ \frac{\alpha_{23} \alpha_{31} - \alpha_{21} \alpha_{33}}{-\alpha_{13} \alpha_{22} \alpha_{31} + \alpha_{12} \alpha_{23} \alpha_{31} + \alpha_{13} \alpha_{21} \alpha_{32} - \alpha_{11} \alpha_{23} \alpha_{32} - \alpha_{12} \alpha_{21} \alpha_{33} + \alpha_{11} \alpha_{22} \alpha_{33}} \\ \frac{\alpha_{22} \alpha_{31} - \alpha_{21} \alpha_{32}}{\alpha_{13} \alpha_{22} \alpha_{31} - \alpha_{12} \alpha_{23} \alpha_{31} - \alpha_{13} \alpha_{21} \alpha_{32} + \alpha_{11} \alpha_{23} \alpha_{32} + \alpha_{12} \alpha_{21} \alpha_{33} - \alpha_{11} \alpha_{22} \alpha_{33}} \\ \frac{1}{\sigma_1} \end{array}$$

$(\gamma_1 // ds[1] // ds[2] // ds[3]) = \gamma_1^{-1}$ // Simplify

$$- \frac{1}{\sigma_1 \sigma_2 \sigma_3} \omega_1 (\alpha_{13} (\alpha_{22} \alpha_{31} - \alpha_{21} \alpha_{32}) + \alpha_{12} (-\alpha_{23} \alpha_{31} + \alpha_{21} \alpha_{33}) + \alpha_{11} (\alpha_{23} \alpha_{32} - \alpha_{22} \alpha_{33})) = \frac{1}{\omega_1}$$

Other

R3

{ Xm_{51} Xm_{62} Xp_{34} // Γ // $dm[1, 4, 1]$ // $dm[2, 5, 2]$ // $dm[3, 6, 3]$,
 Xp_{61} Xm_{24} Xm_{35} // Γ // $dm[1, 4, 1]$ // $dm[2, 5, 2]$ // $dm[3, 6, 3]$ }

R3

$$\left\{ \begin{array}{c} \left(\begin{array}{cccc} 1 & S_1 & S_2 & S_3 \\ S_1 & \frac{T_3}{T_2} & 0 & 0 \\ S_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ S_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{array} \right), \left(\begin{array}{cccc} 1 & S_1 & S_2 & S_3 \\ S_1 & \frac{T_3}{T_2} & 0 & 0 \\ S_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ S_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{array} \right) \end{array} \right\}$$

$$\gamma = \text{Xm}_{12,1} \text{Xm}_{27} \text{Xm}_{83} \text{Xm}_{4,11} \text{Xp}_{16,5} \text{Xp}_{6,13} \text{Xp}_{14,9} \text{Xp}_{10,15} // \Gamma$$

$$\begin{pmatrix} 1 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_1 & \frac{1}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_2 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_3 & 0 & 0 & \frac{1}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+T_4}{T_4} & 0 & 0 & 0 & 0 & 0 \\ s_5 & 0 & 0 & 0 & 0 & T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - T_6 & 0 & 0 & 0 \\ s_7 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_8 & 0 & 0 & \frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 - T_{10} & 0 \\ s_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_4} & 0 & 0 & 0 & 0 & 0 \\ s_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ s_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & 0 & 0 \\ s_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - T_{14} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ s_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{10} & 0 \\ s_{16} & 0 & 0 & 0 & 0 & 1 - T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \Sigma & \frac{1}{T_{12}} & 1 & \frac{1}{T_8} & 1 & T_{16} & 1 & \frac{1}{T_2} & 1 & T_{14} & 1 & \frac{1}{T_4} & 1 & T_6 & 1 & T_{10} & 1 \end{pmatrix}$$

Do[$\gamma = \gamma // \text{dm}_{1k \rightarrow 1}, \{k, 2, 10\}$]; γ

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2} & s_1 & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_1 & \frac{T_{14} (-T_1+T_1^2+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & \frac{(-1+T_1) (1-T_1+T_1^2) T_{14} T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & 0 & -\frac{(-1+T_1) (1-T_1+T_1^2) T_{14}}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 1 - T_1 & 0 \\ s_{11} & 0 & \frac{1}{T_1} & 0 & 0 & 0 & 0 & 0 \\ s_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 1 & 0 & 0 & 0 & 0 \\ s_{13} & 0 & 0 & 0 & T_1 & 0 & 0 & 0 \\ s_{14} & -\frac{(-1+T_{14}) (-T_1+T_1^2+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1) (1-T_1+T_1^2) (-1+T_{14}) T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & 0 & \frac{(-1+T_1) (1-T_1+T_1^2) (-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}} & 1 & 0 & 0 \\ s_{15} & 0 & 0 & 0 & 0 & 0 & T_1 & 0 \\ s_{16} & -\frac{T_1 (-1+T_{16})}{T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & \frac{(-1+T_1)^2 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 0 & 1 \\ \Sigma & \frac{T_{14} T_{16}}{T_1^2 T_{12}} & \frac{1}{T_1} & 1 & T_1 & 1 & T_1 & 1 \end{pmatrix}$$

8_17

$\gamma = \text{Xm}_{12,1} \text{Xm}_{27} \text{Xm}_{83} \text{Xm}_{4,11} \text{Xp}_{16,5} \text{Xp}_{6,13} \text{Xp}_{14,9} \text{Xp}_{10,15} // \Gamma$;
Do[$\gamma = \gamma // \text{dm}[1, k, 1], \{k, 2, 16\}$]; γ

8_17

$$\begin{pmatrix} -\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3} & s_1 \\ s_1 & 1 \\ \Sigma & 1 \end{pmatrix}$$

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links'.

$$\begin{pmatrix} (1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9) (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9) & & & & \\ & T_1 T_5 T_9 & & & \\ & & S_1 & & \\ & & & S_5 & \\ & & & & S_9 \\ & & & & & \Sigma \\ & & & & & & 1 \end{pmatrix}$$

```
MVA[Γ, Link["L6a4"]]
```

$$-\frac{(-1+T_1)(-1+T_5)(-1+T_9)}{T_1 T_5}$$

```
Factor[ $\frac{1}{MVA[\Gamma, \#]}$  (MultivariableAlexander[#][T] /. T[i_] => T_skeleton[#][[i,1]]) & /@ AllLinks[{2, 8}]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links'.

$$\left\{ -T_1^2 T_3, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} T_5^{3/2}, -T_1^{3/2} \sqrt{T_5}, -T_1^2 T_7^2, -T_1^2 T_7, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2}, \right. \\ \left. -\frac{\sqrt{T_1} \sqrt{T_5}}{T_9^{3/2}}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^{7/2}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -\frac{\sqrt{T_1}}{T_5^{3/2}}, -T_1 T_7^2, -\frac{1}{T_7}, -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}}, -T_1^{3/2} T_5^{7/2}, \right. \\ \left. -\sqrt{T_1} T_5^{5/2}, -\sqrt{T_1} T_5^{3/2}, -\frac{\sqrt{T_1}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} T_5^{3/2}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -\frac{T_1^{3/2}}{\sqrt{T_5}}, -T_1^{3/2} T_5^{7/2}, -\frac{T_1}{T_7}, \right. \\ \left. -T_1 T_7, -T_1^2 T_7^3, -T_1^2 T_7^3, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{5/2} T_9^{5/2}, -T_1^{3/2} T_5^{3/2} \sqrt{T_9}, -\sqrt{T_1}, -T_1^{3/2} T_5^2 T_{11}^2, \right. \\ \left. -\frac{\sqrt{T_1}}{T_{11}^2}, -\frac{\sqrt{T_1} T_5}{T_{11}}, -\frac{T_1^{3/2} \sqrt{T_5}}{T_{13}^{3/2}}, -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2}, -T_1^{3/2} T_5^{3/2}, -\sqrt{T_1} \sqrt{T_5}, -T_1^{3/2} T_5^2 T_{11}^2, \right. \\ \left. -\sqrt{T_1} T_5^2 T_{11}, -\sqrt{T_1} T_5^{3/2} T_{11}^{3/2}, -T_1^{3/2} T_5^{5/2} \sqrt{T_{13}}, -\frac{\sqrt{T_1} \sqrt{T_5}}{\sqrt{T_9} \sqrt{T_{13}}}, -\sqrt{T_1} \sqrt{T_5} \sqrt{T_9} \sqrt{T_{13}} \right\}$$

α ↔ Γ Conversions

```
{Xp[1, 2] // Γ, Xp[1, 2] // A // Γ}
```

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix} \right\}$$

`{Xm[1, 2] // A, Xm[1, 2] // Γ // A}`

$$\left\{ \begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1} (1-e^{c_1})}{c_1} \end{pmatrix}, \begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1} (1-e^{c_1})}{c_1} \end{pmatrix} \right\}$$

`Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];`

$$\gamma_0 = \Gamma \left[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\} \right];$$

`{γ0, γ0 // A, γ0 // A // Γ, (γ0 // A // Γ) /. {α → 1 - γ - φ, δ → 1 - β - ψ, Ξ → 1 - θ - ε}}`

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \frac{-\alpha + \sigma_a}{c_a} & -\frac{\beta}{c_a} & -\frac{\theta}{c_a} \\ t[b] & -\frac{\gamma}{c_b} & \frac{-\delta + \sigma_b}{c_b} & -\frac{\epsilon}{c_b} \\ t[S] & -\frac{\phi}{c_s} & -\frac{\psi}{c_s} & \frac{-\Xi + \sigma_s}{c_s} \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & 1 - \gamma - \phi & \beta & \theta \\ s_b & \gamma & 1 - \beta - \psi & \epsilon \\ s_s & \phi & \psi & 1 - \epsilon - \theta \\ \Sigma & 1 - \alpha - \gamma - \phi + \sigma_a & 1 - \beta - \delta - \psi + \sigma_b & 1 - \epsilon - \theta - \Xi + \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & 1 - \gamma - \phi & \beta & \theta \\ s_b & \gamma & 1 - \beta - \psi & \epsilon \\ s_s & \phi & \psi & 1 - \epsilon - \theta \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix} \right\}$$

`Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];`

$$\gamma_0 = \Gamma \left[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\} \right];$$

`{γ0 // dm[a, b, c] // A, γ0 // A // dm[a, b, c]} /. {α → 1 - γ - φ, δ → 1 - β - ψ, Ξ → 1 - θ - ε}`

$$\left\{ \begin{pmatrix} \omega - \beta \omega & h[c] & h[S] \\ t[c] & \frac{1 - \beta - \phi + \beta \phi - \psi + \gamma \psi + \phi \psi - \sigma_a \sigma_b + \beta \sigma_a \sigma_b}{-c_c + \beta c_c} & \frac{\epsilon - \beta \epsilon + \theta - \beta \theta - \theta \psi}{-c_c + \beta c_c} \\ t[S] & \frac{\phi - \beta \phi + \psi - \gamma \psi - \phi \psi}{-c_s + \beta c_s} & \frac{1 - \beta - \epsilon + \beta \epsilon - \theta + \beta \theta + \theta \psi - \sigma_s + \beta \sigma_s}{-c_s + \beta c_s} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \omega - \beta \omega & h[c] & h[S] \\ t[c] & \frac{1 - \beta - \phi + \beta \phi - \psi + \gamma \psi + \phi \psi - \sigma_a \sigma_b + \beta \sigma_a \sigma_b}{-c_c + \beta c_c} & \frac{\epsilon - \beta \epsilon + \theta - \beta \theta - \theta \psi}{-c_c + \beta c_c} \\ t[S] & \frac{\phi - \beta \phi + \psi - \gamma \psi - \phi \psi}{-c_s + \beta c_s} & \frac{1 - \beta - \epsilon + \beta \epsilon - \theta + \beta \theta + \theta \psi - \sigma_s + \beta \sigma_s}{-c_s + \beta c_s} \end{pmatrix} \right\}$$

The KV solution in Γ , starting from α

V // A // α Collect[FullSimplify]

$$\left(\begin{array}{l} 2^{1/4} \left(\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1} \right)^{1/4} \left(\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2} \right)^{1/4} \\ \frac{\left(\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2} \right)^{1/4}}{h[1]} \\ t[1] \\ t[2] \end{array} \right)$$

$$\frac{-\sqrt{2} \sqrt{\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}} c_2 + 2 \sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} c_2 \sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}{2 \sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} c_1^2 \sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2 \sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} c_1 c_2 \sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}$$

$$\frac{\sqrt{2} \sqrt{\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}} - 2 \sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} \sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}{2 \sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} c_1 \sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2 \sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} c_2 \sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}} - \frac{e^{-\frac{c_1}{2}} c_1 \sqrt{\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}} + e^{\frac{3c_1}{2}+c_2} c_1 \sqrt{s}}{\sqrt{s}}$$

V // A // Γ //

Γ Collect[Assuming[$T_1 > 0$ && $T_2 > 0$, (# /. {Sinh[x_] => $\frac{e^x - e^{-x}}{2}$ }] // FullSimplify] &]

$$\left(\begin{array}{l} \left(\frac{-1+T_1}{\text{Log}[T_1]} \right)^{1/4} \left(\frac{-1+T_2}{\text{Log}[T_2]} \right)^{1/4} \\ \frac{\left(\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]} \right)^{1/4}}{S_1} \\ S_1 \\ S_2 \\ \Sigma \end{array} \right)$$

$$\frac{\text{Log}[T_1] \left(\text{Log}[T_2] \sqrt{\frac{-1+T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} - \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} + T_1 \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \right)}{\text{Log}[T_1 T_2] (-1+T_1) \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}$$

$$\frac{\text{Log}[T_1] \left(\sqrt{\frac{-1+T_1}{\text{Log}[T_1]}} T_2 - \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \right)}{\text{Log}[T_1 T_2] \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}$$

$$\frac{\text{Log}[T_2] \left(-T_1 \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} + \sqrt{\frac{-1+T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \right)}{\text{Log}[T_1 T_2] \sqrt{\frac{-1+T_1}{\text{Log}[T_1]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}$$

$$\frac{\text{Log}[T_1] \left(\sqrt{\frac{-1+T_1}{\text{Log}[T_1]}} T_2 + \text{Log}[T_2] \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}} \right)}{\text{Log}[T_1 T_2] \sqrt{\frac{-1+T_2}{\text{Log}[T_2]}} \sqrt{\frac{-1+T_1 T_2}{\text{Log}[T_1 T_2]}}}$$

$$\frac{1}{\sqrt{T_1}}$$

Γ Simp = Assuming[$T_1 > 1$ && $T_2 > 1$, (# /. {Sinh[x_] => $\frac{e^x - e^{-x}}{2}$ }] // FullSimplify] &;

V // A // Γ

$$\left(\begin{array}{l} \left(\frac{\text{Log}[T_1 T_2] (-1+T_1) (-1+T_2)}{\text{Log}[T_1] \text{Log}[T_2] (-1+T_1 T_2)} \right)^{1/4} \\ S_1 \\ S_2 \\ \Sigma \end{array} \right)$$

$$\frac{\text{Log}[T_1] + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] T_1 (-1+T_2)}{(-1+T_1) (-1+T_1 T_2)}}}{\text{Log}[T_1 T_2]}$$

$$\frac{\text{Log}[T_1] \left(-1+T_2 \left(T_1 - \sqrt{\frac{\text{Log}[T_2] \text{Log}[T_1 T_2]}{\text{Log}[T_1 T_2]}} \right) \right)}{\text{Log}[T_1 T_2] (-1+T_1)}$$

$$\frac{\text{Log}[T_2] \left(-1 + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_1 T_2] T_1 (-1+T_2)}{\text{Log}[T_2] (-1+T_1) (-1+T_1 T_2)}} \right)}{\text{Log}[T_1 T_2]}$$

$$-\frac{-\text{Log}[T_2] + T_2 \left(\text{Log}[T_2] T_1 + \sqrt{\frac{\text{Log}[T_1]^3 \text{Log}[T_2] (-1+T_1) T_1 (-1+T_2)}{\text{Log}[T_1 T_2] (-1+T_2)}} \right)}{\text{Log}[T_1 T_2] (-1+T_1)}$$

$$\frac{1}{\sqrt{T_1}}$$

$\Gamma[V] ** \Gamma[Vi]$

$$\begin{matrix} 1 \\ S_1 \\ S_2 \\ \Sigma \end{matrix} \left(\frac{1}{\begin{matrix} S_1 \\ S_2 \\ \Sigma \end{matrix}} \right)$$

$$\frac{1}{\text{Log}[T_1 T_2] (-1 + T_1 T_2)}$$

$$\left(-\text{Log}[T_1 T_2] + T_2 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_1) T_1 (-1 + T_2)}{-1 + T_1 T_2}} + T_1 \left(\text{Log}[T_1 T_2] + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_2)}{(-1 + T_1) T_1 (-1 + T_1 T_2)}} - T_1 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_1) T_1 (-1 + T_2)}{(-1 + T_1) T_1 (-1 + T_1 T_2)}} \right) \right) // \text{PowerExpand} // \text{Simplify}$$

1

$$\left((-1 + T_2) \left(-(-1 + T_1) T_1 \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_2)}{(-1 + T_1) T_1 (-1 + T_1 T_2)}} + \sqrt{\frac{\text{Log}[T_1] \text{Log}[T_2] \text{Log}[T_1 T_2] (-1 + T_1) T_1 (-1 + T_2)}{-1 + T_1 T_2}} \right) \right) // \text{PowerExpand} // \text{Simplify}$$

0

dS and dA for Γ , starting from α

```
Clear[α, θ, φ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . {α θ, φ Ξ} . {h_a, h_s}];
((γ0 // A // dS[a] // Γ) == (γ0 // A // dA[a] // Γ)) /. {α -> 1 - φ, Ξ -> 1 - θ}
True
```

`(γ0 // A // dS[a] // Γ)`

$$\begin{pmatrix} \frac{(-1+\phi)\omega}{-1+\alpha+\phi-\sigma_a} & S_a & S_S \\ S_a & -\frac{1}{-1+\phi} & -\frac{\theta}{-1+\phi} \\ S_S & \frac{\phi}{-1+\phi} & \frac{-1+\theta+\phi}{-1+\phi} \\ \Sigma & -\frac{1}{-1+\alpha+\phi-\sigma_a} & -\frac{1-\alpha-\theta+\alpha\theta-\Xi+\alpha\Xi-\phi+\theta\phi+\Xi\phi+\sigma_a-\theta\sigma_a-\Xi\sigma_a+\sigma_S-\alpha\sigma_S-\phi\sigma_S+\sigma_a\sigma_S}{-1+\alpha+\phi-\sigma_a} \end{pmatrix}$$

`(γ0 // A // dA[a] // Γ) /. {φ → 1 - α, Ξ → 1 - θ} // ΓCollect`

$$\begin{pmatrix} \frac{\alpha\omega}{\sigma_a} & S_a & S_S \\ S_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ S_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}$$

`(γ0 // dA[a]) /. {φ → 1 - α, Ξ → 1 - θ} // ΓCollect`

$$\begin{pmatrix} \frac{\alpha\omega}{\sigma_a} & S_a & S_S \\ S_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ S_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}$$

`(γ0 // A // dA[a] // Γ) == (γ0 // dA[a]) /. {φ → 1 - α, Ξ → 1 - θ} // Simplify`

True

dΔ for Γ, starting from α

`Clear[α, θ, φ, Ξ, ω];`

`γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . {α θ / φ Ξ} . {h_a, h_s}];`

`((γ0 // A // dΔ[a, b, c] // Γ) /. {α → 1 - φ, Ξ → 1 - θ}) // ΓCollect`

$$\begin{pmatrix} \omega & S_b & S_c & S_S \\ S_b & -\frac{-\text{Log}[T_b]+\phi\text{Log}[T_b]-\text{Log}[T_c]\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & -\frac{\text{Log}[T_b](-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\text{Log}[T_b]}{\text{Log}[T_b]+\text{Log}[T_c]} \\ S_c & -\frac{\text{Log}[T_c](-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\text{Log}[T_c]-\phi\text{Log}[T_c]+\text{Log}[T_b]\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\text{Log}[T_c]}{\text{Log}[T_b]+\text{Log}[T_c]} \\ S_S & \phi & \phi & 1-\theta \\ \Sigma & \sigma_a & \sigma_a & \sigma_S \end{pmatrix}$$

```

Clear[α, θ, φ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . (α θ / φ Ξ) . {h_a, h_s}];
γ0 // A // dΔ[a, b, c] // dS[c] // dm[b, c, a]

Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>

Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity +  $\frac{(\Xi - \theta \phi - \Xi \phi - \sigma_s + \phi \sigma_s) t[S]}{-c_s + \phi c_s}$  encountered. >>

(  $\frac{-\omega + \phi \omega}{-1 + \alpha \phi - \sigma_a}$  )
    
```

qΔ for Γ, starting from α

```

Clear[α, θ, φ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . (α θ / φ Ξ) . {h_a, h_s}];
γ0 // A // qΔ[a, b, c] // Γ

( ω
  S_b -  $\frac{1 - \alpha - \phi + \alpha T_c - T_b T_c + \phi T_b T_c + \sigma_a - T_c \sigma_a}{-1 + T_b T_c}$ 
  S_c  $\frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{-1 + T_b T_c}$ 
  S_s  $\frac{(-1 + T_c) (\alpha - \sigma_a)}{-1 + T_b T_c}$ 
  Σ  $1 - \alpha - \phi + \sigma_a$ 
  S_c  $\frac{1 - \phi - \alpha T_c - T_b T_c + \alpha T_b T_c + \phi T_b T_c + T_c \sigma_a - T_b T_c \sigma_a}{-1 + T_b T_c}$ 
  S_s φ
  Σ  $1 - \alpha - \phi + \sigma_a$ 
  θ (-)
  -
  θ
  -
  :
```

```

Clear[α, θ, φ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . (α θ / φ Ξ) . {h_a, h_s}];
γ0 // qΔ[a, b, c]

( ω
  S_b  $\frac{-\alpha T_c + \alpha T_b T_c - \sigma_a + T_c \sigma_a}{-1 + T_b T_c}$ 
  S_c  $\frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{-1 + T_b T_c}$ 
  S_s  $\frac{\theta (-1 + T_b) T_c}{-1 + T_b T_c}$ 
  Σ σ_a
  S_c  $\frac{(-1 + T_c) (\alpha - \sigma_a)}{-1 + T_b T_c}$ 
  S_s  $\frac{-\alpha + \alpha T_c - T_c \sigma_a + T_b T_c \sigma_a}{-1 + T_b T_c}$ 
  Σ σ_a
  S_s φ
  Σ σ_a
  θ (-1 + T_c)
  -1 + T_b T_c
  Ξ
  S_s
```

```

Clear[α, θ, φ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_s σ_s, {t_a, t_s} . (α θ / φ Ξ) . {h_a, h_s}];
Simplify[
  (γ0 // qΔ[a, b, c]) [[3]] == (γ0 // A // qΔ[a, b, c] // Γ) [[3]] /. {α -> 1 - φ, θ -> 1 - Ξ}
]
True
    
```

q Δ tests for Γ

$$\{t1 = Xp_{13} // \Gamma // q\Delta[1, 1, 2], t2 = (\epsilon[1] Xp_{23} // \Gamma) ** (\epsilon[2] Xp_{13} // \Gamma), t1 == t2\}$$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1 + T_1) T_2 \\ s_2 & 0 & 1 & 1 - T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Sigma & 1 & 1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1 + T_1) T_2 \\ s_2 & 0 & 1 & 1 - T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Sigma & 1 & 1 & T_1 T_2 \end{pmatrix}, \text{True} \right\}$$

$$\{t1 = Xm_{13} // \Gamma // q\Delta[1, 1, 2], t2 = (\epsilon[2] Xm_{13} // \Gamma) ** (\epsilon[1] Xm_{23} // \Gamma), t1 == t2\}$$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1+T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Sigma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1+T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Sigma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \text{True} \right\}$$

$$\{t1 = Xp_{3,1} // \Gamma // q\Delta[1, 1, 2], t2 = (\epsilon[1] Xp_{3,2} // \Gamma) ** (\epsilon[2] Xp_{3,1} // \Gamma), t1 == t2\}$$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1 - T_3 & 1 - T_3 & 1 \\ \Sigma & T_3 & T_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1 - T_3 & 1 - T_3 & 1 \\ \Sigma & T_3 & T_3 & 1 \end{pmatrix}, \text{True} \right\}$$

$$\{t1 = Xm_{3,1} // \Gamma // q\Delta[1, 1, 2], t2 = (\epsilon[2] Xm_{3,1} // \Gamma) ** (\epsilon[1] Xm_{3,2} // \Gamma), t1 == t2\}$$

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{pmatrix}, \text{True} \right\}$$

Clear[$\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega$];

$$\{\gamma 0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}], \gamma 0 // dm[a, b, c]\}$$

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} -(-1 + \beta) \omega & s_c & s_s \\ s_c & \frac{-\gamma + \beta \gamma - \alpha \delta}{-1 + \beta} & \frac{-\epsilon + \beta \epsilon - \delta \theta}{-1 + \beta} \\ s_s & \frac{-\phi + \beta \phi - \alpha \psi}{-1 + \beta} & \frac{-\Xi + \beta \Xi - \theta \psi}{-1 + \beta} \\ \Sigma & \sigma_a \sigma_b & \sigma_s \end{pmatrix} \right\}$$

`Clear[α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ, ω];`

$$\{\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}],$$

`t1 = γ0 // dm[a, b, c] // dS[c], t2 = γ0 // dS[a] // dS[b] // dm[b, a, c], t1 == t2}`

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{pmatrix}, \text{True} \right\}$$

`Clear[α, θ, φ, Ξ, ω];`

$$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_s \sigma_s, \{t_a, t_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{h_a, h_s\}];$$

`FullSimplify[{1, 1}.dS[a][γ0][A], And@@Thread[{1, 1}.γ0[A] == {1, 1}]]`

$$\left\{ \frac{1-\phi}{\alpha}, \frac{1-\phi}{\alpha} \right\}$$

`{1, 1}.dS[a][γ0][A] // Simplify`

$$\left\{ \frac{1-\phi}{\alpha}, \frac{\theta+\alpha\Xi-\theta\phi}{\alpha} \right\}$$

`And@@Thread[{1, 1}.γ0[A] == {1, 1}]`

$$\alpha + \phi == 1 \ \&\& \ \theta + \Xi == 1$$

`Clear[α, θ, φ, Ξ, ω];`

$$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_s \sigma_s, \{t_a, t_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} \cdot \{h_a, h_s\}];$$

`γ0 // dS[a] // dS[a]`

$$\begin{pmatrix} \omega & s_a & s_s \\ s_a & \alpha & \theta \\ s_s & \phi & \Xi \\ \Sigma & \sigma_a & \sigma_s \end{pmatrix}$$

`Clear[α, θ, φ, Ξ, ω];`

`γ0 = Γ[ω, ha σa + hs σs, {ta, ts}. (α[Ta] θ[Ta]
φ[Ta] Ξ[Ta]). {ha, hs}]`;

`{(γ0 // dη[a]) (ε[a] // Γ), γ0 // qΔ[a, b, c] // dS[c] // dm[b, c, a],
γ0 // qΔ[a, b, c] // dS[c] // dm[c, b, a],
γ0 // qΔ[a, b, c] // dS[b] // dm[b, c, a],
γ0 // qΔ[a, b, c] // dS[b] // dm[c, b, a]}`

$$\left\{ \begin{pmatrix} \omega & s_a & s_s \\ s_a & 1 & 0 \\ s_s & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_s \end{pmatrix}, \begin{pmatrix} \frac{\omega \alpha[1]}{\sigma_a} & s_a & s_s \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_s & 0 & \frac{\alpha[1] \Xi[1] - \theta[1] \phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_s \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} \omega & s_a & s_s \\ s_a & 1 & \theta[1] \\ s_s & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_s \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_s \\ s_a & 1 & \theta[1] \\ s_s & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_s \end{pmatrix}, \begin{pmatrix} \frac{\omega \alpha[1]}{\sigma_a} & s_a & s_s \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_s & 0 & \frac{\alpha[1] \Xi[1] - \theta[1] \phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_s \end{pmatrix} \right\}$$

`{Xp[S, a] // Γ, Xp[S, a] // Γ // qΔ[a, b, c] // dS[c] // dm[c, b, a]}`

$$\left\{ \begin{pmatrix} 1 & s_a & s_s \\ s_a & T_s & 0 \\ s_s & 1 - T_s & 1 \\ \Sigma & T_s & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_s \\ s_a & 1 & 0 \\ s_s & 0 & 1 \\ \Sigma & 1 & 1 \end{pmatrix} \right\}$$

`Clear[α, θ, φ, Ξ, ω];`

`γ0 = Γ[ω, ha σa + hs σs, {ta, ts}. (α θ
φ Ξ). {ha, hs}]`;

`{t1 = γ0 // qΔ[a, b, c] // dS[b] // dS[c],
t2 = γ0 // dS[a] // qΔ[a, c, b], Simplify[t1 == t2]}`

$$\left\{ \begin{pmatrix} \frac{\alpha \omega}{\sigma_a} & s_b & s_c & s_s \\ s_b & \frac{-\alpha T_b + \alpha T_b T_c - \sigma_a + T_b \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & -\frac{(-1 + T_b) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta (-1 + T_b)}{\alpha (-1 + T_b T_c)} \\ s_c & -\frac{T_b (-1 + T_c) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{-\alpha + \alpha T_b - T_b \sigma_a + T_b T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta T_b (-1 + T_c)}{\alpha (-1 + T_b T_c)} \\ s_s & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha \Xi - \theta \phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_s \end{pmatrix},$$

$$\left\{ \begin{pmatrix} \frac{\alpha \omega}{\sigma_a} & s_b & s_c & s_s \\ s_b & \frac{-\alpha T_b + \alpha T_b T_c - \sigma_a + T_b \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & -\frac{(-1 + T_b) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta (-1 + T_b)}{\alpha (-1 + T_b T_c)} \\ s_c & -\frac{T_b (-1 + T_c) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{-\alpha + \alpha T_b - T_b \sigma_a + T_b T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta T_b (-1 + T_c)}{\alpha (-1 + T_b T_c)} \\ s_s & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha \Xi - \theta \phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_s \end{pmatrix}, \text{True} \right\}$$

dA tests for Γ

`{Xp[1, 2] // Γ , (Xm[1, 2] // Γ) /. T1 -> 1 / T1, Xm[1, 2] // Γ // dA[1] // dA[2]}`

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix} \right\}$$

`{Xm[1, 2] // Γ , (Xp[1, 2] // Γ) /. T1 -> 1 / T1, Xp[1, 2] // Γ // dA[1] // dA[2]}`

$$\left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix} \right\}$$

`Clear[$\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega$];`

$$\{\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_b \sigma_b + h_s \sigma_s, \{t_a, t_b, t_s\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{h_a, h_b, h_s\}],$$

`t1 = γ_0 // dm[a, b, c] // dA[c], t2 = γ_0 // dA[a] // dA[b] // dm[b, a, c], t1 == t2}`

$$\left\{ \begin{pmatrix} \omega & s_a & s_b & s_s \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_s & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_s \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\omega}{\sigma_a\sigma_b} & s_c & s_s \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_s & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\sigma_b} & \sigma_s \end{pmatrix}, \text{True} \right\}$$

Clear[α, θ, ϕ, Σ, ω];

$\gamma_0 = \Gamma[\omega, h_a \sigma_a + h_s \sigma_s, \{t_a, t_s\} \cdot \begin{pmatrix} \alpha & \theta \\ \phi & \Sigma \end{pmatrix} \cdot \{h_a, h_s\}];$

{t1 = γ0 // qΔ[a, b, c] // dA[b] // dA[c],

t2 = γ0 // dA[a] // qΔ[a, b, c], Simplify[t1 == t2]}

$$\begin{pmatrix} \frac{\alpha \omega}{\sigma_a} & S_b & S_c & S_s \\ S_b & \frac{-\alpha + \alpha T_c - T_c \sigma_a + T_b T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & -\frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta (-1 + T_b) T_c}{\alpha (-1 + T_b T_c)} \\ S_c & -\frac{(-1 + T_c) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{-\alpha T_c + \alpha T_b T_c - \sigma_a + T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta (-1 + T_c)}{\alpha (-1 + T_b T_c)} \\ S_s & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha \Sigma - \theta \phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_s \end{pmatrix},$$

$$\begin{pmatrix} \frac{\alpha \omega}{\sigma_a} & S_b & S_c & S_s \\ S_b & \frac{-\alpha + \alpha T_c - T_c \sigma_a + T_b T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & -\frac{(-1 + T_b) T_c (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta (-1 + T_b) T_c}{\alpha (-1 + T_b T_c)} \\ S_c & -\frac{(-1 + T_c) (\alpha - \sigma_a)}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{-\alpha T_c + \alpha T_b T_c - \sigma_a + T_c \sigma_a}{\alpha (-1 + T_b T_c) \sigma_a} & \frac{\theta (-1 + T_c)}{\alpha (-1 + T_b T_c)} \\ S_s & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha \Sigma - \theta \phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_s \end{pmatrix}, \text{True}$$

$n = 4; \gamma_0 = \Gamma\left[\omega, \sum_{a=0}^n h_a \sigma_a, \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{10 a+b}\right]$

$$\begin{pmatrix} \omega & S_1 & S_2 & S_3 & S_4 \\ S_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ S_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ S_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ S_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

γ0 // dA[1] // dA[2] // dA[3] // dA[4]

$$\left(\frac{\omega (\alpha_{14} \alpha_{23} \alpha_{32} \alpha_{41} - \alpha_{13} \alpha_{24} \alpha_{32} \alpha_{41} - \alpha_{14} \alpha_{22} \alpha_{33} \alpha_{41} + \alpha_{12} \alpha_{24} \alpha_{33} \alpha_{41} + \alpha_{13} \alpha_{22} \alpha_{34} \alpha_{41} - \alpha_{12} \alpha_{23} \alpha_{34} \alpha_{41} - \alpha_{14} \alpha_{23} \alpha_{31} \alpha_{42} + \alpha_{13} \alpha_{24} \alpha_{31} \alpha_{42} + \alpha_{14} \alpha_{21} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{33} \alpha_{42})}{\dots} \right)$$

(γ0 // dA[1] // dA[2] // dA[3] // dA[4]) ** γ0

$$\left(\frac{\omega^2 (\alpha_{14} \alpha_{23} \alpha_{32} \alpha_{41} - \alpha_{13} \alpha_{24} \alpha_{32} \alpha_{41} - \alpha_{14} \alpha_{22} \alpha_{33} \alpha_{41} + \alpha_{12} \alpha_{24} \alpha_{33} \alpha_{41} + \alpha_{13} \alpha_{22} \alpha_{34} \alpha_{41} - \alpha_{12} \alpha_{23} \alpha_{34} \alpha_{41} - \alpha_{14} \alpha_{23} \alpha_{31} \alpha_{42} + \alpha_{13} \alpha_{24} \alpha_{31} \alpha_{42} + \alpha_{14} \alpha_{21} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{23} \alpha_{33} \alpha_{42})}{\dots} \right)$$

$$\begin{aligned}
 & (\alpha_{14} \alpha_{23} \alpha_{32} \alpha_{41} - \alpha_{13} \alpha_{24} \alpha_{32} \alpha_{41} - \alpha_{14} \alpha_{22} \alpha_{33} \alpha_{41} + \alpha_{12} \alpha_{24} \alpha_{33} \alpha_{41} + \alpha_{13} \alpha_{22} \alpha_{34} \alpha_{41} - \alpha_{12} \alpha_{23} \alpha_{34} \alpha_{41} - \\
 & \alpha_{14} \alpha_{23} \alpha_{31} \alpha_{42} + \alpha_{13} \alpha_{24} \alpha_{31} \alpha_{42} + \alpha_{14} \alpha_{21} \alpha_{33} \alpha_{42} - \alpha_{11} \alpha_{24} \alpha_{33} \alpha_{42} - \alpha_{13} \alpha_{21} \alpha_{34} \alpha_{42} + \\
 & \alpha_{11} \alpha_{23} \alpha_{34} \alpha_{42} + \alpha_{14} \alpha_{22} \alpha_{31} \alpha_{43} - \alpha_{12} \alpha_{24} \alpha_{31} \alpha_{43} - \alpha_{14} \alpha_{21} \alpha_{32} \alpha_{43} + \alpha_{11} \alpha_{24} \alpha_{32} \alpha_{43} + \\
 & \alpha_{12} \alpha_{21} \alpha_{34} \alpha_{43} - \alpha_{11} \alpha_{22} \alpha_{34} \alpha_{43} - \alpha_{13} \alpha_{22} \alpha_{31} \alpha_{44} + \alpha_{12} \alpha_{23} \alpha_{31} \alpha_{44} + \alpha_{13} \alpha_{21} \alpha_{32} \alpha_{44} - \\
 & \alpha_{11} \alpha_{23} \alpha_{32} \alpha_{44} - \alpha_{12} \alpha_{21} \alpha_{33} \alpha_{44} + \alpha_{11} \alpha_{22} \alpha_{33} \alpha_{44}) == \text{Det} \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \right]
 \end{aligned}$$

True

⊖ tests

⊖[1, 2] // A

$$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-c_1 + e^{\frac{c_2}{2}} c_1 - e^{\frac{c_1 + c_2}{2}} c_2 + e^{\frac{c_2}{2}} c_2}{c_1^2 + c_1 c_2} & \frac{-1 + e^{\frac{c_1 + c_2}{2}}}{c_1 + c_2} \\ t[2] & \frac{-1 + e^{\frac{c_1 + c_2}{2}}}{c_1 + c_2} & \frac{e^{\frac{c_1}{2}} c_1 - e^{\frac{c_1 + c_2}{2}} c_1 - c_2 + e^{\frac{c_1}{2}} c_2}{c_1 c_2 + c_2^2} \end{pmatrix}$$

(V // A) ** (⊖[1, 2] // A)

$$\begin{pmatrix} 2^{1/4} \left(\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1} \right)^{1/4} \left(\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2} \right)^{1/4} & h[1] \\ \frac{\left(\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2} \right)^{1/4}}{t[1]} & \frac{-\sqrt{2} e^{\frac{c_1 + c_2}{2}} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} c_2 - 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}} + 2 e^{\frac{c_2}{2}} \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}} + 2 e^{\frac{c_2}{2}} \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1^2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2^2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}}}{\sqrt{2} e^{\frac{c_1 + c_2}{2}} \sqrt{\frac{\sinh\left[\frac{c_2}{2}\right]}{c_2}} - 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}} - 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}}}{2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_1 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}} + 2 \sqrt{\frac{\sinh\left[\frac{c_1}{2}\right]}{c_1}} c_2 \sqrt{\frac{\sinh\left[\frac{1}{2}(c_1 + c_2)\right]}{c_1 + c_2}}}$$


```
( $\Theta[1, 2]$  // A) == (Vi // A) ** (Xp[1, 2] // A) ** (V // A //  $d\sigma[1 \rightarrow 2, 2 \rightarrow 1]$ ) // Simplify
```

```
Simplify::time :
```

```
Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of
TimeConstraint option may improve the result of simplification. >>
```

```
Simplify::time :
```

```
Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of
TimeConstraint option may improve the result of simplification. >>
```

```
Simplify::time :
```

```
Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of
TimeConstraint option may improve the result of simplification. >>
```

```
General::stop : Further output of Simplify::time will be suppressed during this calculation. >>
```

```
$Aborted
```

```
(Xp[2, 1] // A) == (V // A //  $d\sigma[1 \rightarrow 2, 2 \rightarrow 1]$ ) ** ( $\Theta[1, 2]$  // A) ** (Vi // A) // Simplify
```

```
True
```