Cheat Sheet β

verification at `2014-04/CheatSheetBeta-Verification.nb`, `2014-05/GoodFormulas/Demo.nb`
`http://drorbn.net/AcademicPensieve/2014-05/`; initiated 24/3/13; continues 2014-04; continued 2014-06; modified 10/6/14, 8:05pm

σ calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $\quad tm^{uv}_w = (T_u, T_v \to T_w)$, $\quad hm^{xy}_z\colon \sigma \mapsto (\sigma\setminus\{x,y\}) \cup (z \to \sigma_x\sigma_y)$, $\quad tha^{ux} = I$, $\quad R^{\pm}_{ux} \mapsto T_u^{\pm 1}$

β-calculus. Constraints. • Sum of column x is $\sigma_x - 1$. • At $T_* = 1$, $\omega = 1$ and $A = 0$.

$$\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array} \underset{\beta}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array} \qquad \begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \Xi\end{array} \xrightarrow[\beta]{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \Xi\end{array}\right)_{T_u,T_v\to T_w} \qquad \begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \Xi\end{array} \xrightarrow[\beta]{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \Xi\end{array}$$

$$\begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \theta\\ T & \phi & \Xi\end{array} \xrightarrow[\substack{\beta\\ \nu:=1+\alpha}]{tha^{ux}} \begin{array}{c|cc}\nu\omega & x & H\\ \hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu\\ T & \phi/\nu & \Xi-\phi\theta/\nu\end{array} \qquad \rho^{\pm}_{ux} \underset{\beta}{=} \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}$$

Gassner calculus. Preserves $C_1 := [\text{col sum} = 1]$ ($\Leftrightarrow$OC) and $C_2 := [\forall a\neq b,\ (T_a-1)\mid A_{ab}]$

$$\left(\begin{array}{c|cc}\nu\omega & c & S\\ \hline c & \beta+\alpha\delta/\nu & \theta+\alpha\epsilon/\nu\\ S & \psi+\delta\phi/\nu & \Xi+\epsilon\phi/\nu\end{array}\right)_{T_a,T_b\to T_c} \xleftarrow[\nu:=1-\gamma]{m^{ba}_c} \begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\theta\psi/\mu\end{array}\right)_{T_a,T_b\to T_c} \qquad R^{\pm}_{ab} \underset{\gamma}{=} \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}$$

$$\begin{array}{c|cc}\omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array} \xrightarrow[\substack{\mu:=T_a-1\\ \nu:=\alpha-\sigma_a}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc}\omega & b & c & S\\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu\\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_aT_a-\nu T_c)/\mu & (T_c-1)\theta/\mu\\ S & \phi & \phi & \Xi\end{array}\right)_{T_a\mapsto T_bT_c}$$

Satisfies: ✓ $R^+_{13}/\!/q\Delta^1_{12} = R^+_{23}\#R^+_{13}$. ✓ $R^-_{13}/\!/q\Delta^1_{12} = R^-_{13}\#R^-_{23}$. ✓ $q\Delta^a_{a_1a_2}/\!/q\Delta^b_{b_1b_2}/\!/m^{a_1b_1}_{c_1}/\!/m^{a_2b_2}_{c_2} = m^{ab}_c/\!/q\Delta^c_{c_1c_2}$.

$$\begin{array}{c|cc}\omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array} \xrightarrow{dS^a} \left(\begin{array}{c|cc}\alpha\omega/\sigma_a & a & S\\ \hline a & \alpha & \theta/\alpha\\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha\end{array}\right)_{T_a\to T_a^{-1}}$$

Satisfies: ✓ $R^{\pm}_{12}/\!/dS^{1\text{ or }2} = R^{\mp}_{12}$. ✓ $dm^{ab}_c/\!/dS^c = dS^a/\!/dS^b/\!/dm^{ba}_c$. ✓ $dS^a/\!/dS^a = I$. ✓ $q\Delta^a_{bc}/\!/dS^b/\!/dS^c = dS^a/\!/q\Delta^a_{cb}$. ✓ Assuming C_2, $d\eta^a/\!/d\epsilon_a = q\Delta^a_{bc}/\!/dS^c/\!/dm^{bc}_a$ (also 3 variants).

The map (tangle $T\mapsto$ matrix A) is anti-multiplicative. The MVA mod units: $L\mapsto(\omega,A)\mapsto\omega\det{}'(A-I)/(1-T')$ ✓

β-better calculus. Constraints. • Sum of column x is $(\sigma_x-1)w$. • $\omega^{k-1}\mid\Lambda^kA$. • At $T_*=1$, $\omega=1$ and $A=0$.

$$\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array} \underset{\beta_b}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & \omega_2A_1 & 0\\ T_2 & 0 & \omega_1A_2\end{array} \qquad \begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \gamma\end{array} \xrightarrow[\beta_b]{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \gamma\end{array}\right)_{T_u,T_v\to T_w} \qquad \rho^{\pm}_{ux} \underset{\beta_b}{=} \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}$$

$$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \gamma\end{array} \xrightarrow[\beta_b]{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \gamma\end{array} \qquad \begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \beta\\ T & \gamma & \delta\end{array} \xrightarrow[\beta_b]{tha^{ux}} \begin{array}{c|cc}\omega+\alpha & x & H\\ \hline u & \sigma_x\alpha & \sigma_x\beta\\ T & \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{array} =: \begin{array}{c|c}\cdot & -\\ \hline | & \begin{pmatrix}\sigma_x & 0\\ 0 & 1\end{pmatrix}\cdot A^{ux}\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\beta_b\ ✓]{m^{ab}_c} \left(\begin{array}{c|cc}\omega+\beta & c & S\\ \hline c & \gamma+\sigma_a\delta+\sigma_b(\alpha+\sigma_a\beta)+\frac{\beta\gamma-\alpha\delta}{\omega} & \epsilon+\sigma_b\theta+\frac{\beta\epsilon-\delta\theta}{\omega}\\ S & \phi+\sigma_a\psi+\frac{\beta\phi-\alpha\psi}{\omega} & \Xi+\frac{\beta\Xi-\psi\theta}{\omega}\end{array}\right)_{T_a,T_b\to T_c}$$

The MVA (mod units): n-component $L\mapsto(\sigma,\omega,A)\mapsto\omega^{2-n}\det{}'(A-\omega\,\mathrm{diag}((\sigma_i-1))/(1-T')$ ✓

Note. $A^{ux} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \frac{(\omega+\alpha)\delta-\gamma\beta}{\omega}\end{pmatrix} = \frac{1}{\omega}\left[(\omega+\alpha)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} - \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix}\right] = \frac{1}{\omega}\left[(\omega+a_{ux})A - a_{*x}a_{u*}\right]$.

Claim. $\omega^{k-1}\mid\Lambda^kA$ and $\omega^k\mid\Lambda^{k+1}A$ implies $(\omega+\alpha)^{k-1}\mid\Lambda^kA^{ux}$, with $\alpha=a_{ux}$.

Proof. With $\bar u\in T^k$ and $\bar x\in H^k$, ω^k divides $\begin{vmatrix}\omega & 0\\ 0 & a_{\bar u\bar x}\end{vmatrix}$ and $\begin{vmatrix}a_{ux} & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix}$ and hence their sum, $\begin{vmatrix}\omega+\alpha & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix} = (\omega+\alpha)\begin{vmatrix}1 & 0\\ 0 & a_{\bar u\bar x}-\frac{1}{\omega+\alpha}a_{\bar ux}a_{u\bar x}\end{vmatrix} = \frac{1}{(\omega+\alpha)^{k-1}}\left|(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right|$. So $\frac{1}{(\omega+\alpha)^{k-1}}\left|\frac{1}{\omega}\left[(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right]\right|$ is integral. □

That is, with $A_{\bar u;\bar x}$ denoting minors, if $\omega^{k-1}\mu_{\bar u;\bar x}=A_{\bar u;\bar x}$ and $\omega^k\mu_{u\bar u;x\bar x}=A_{u\bar u;x\bar x}$, then $(\omega+\alpha)^{k-1}(\mu_{\bar u;\bar x}+\mu_{u\bar u;x\bar x}) = A^{ux}_{\bar u;\bar x}$.

Relations. • $\rho^+_{ux}\rho^-_{vy}/\!/tm^{uv}_w/\!/hm^{xy}_z = t\epsilon_wh\epsilon_z$. • $\rho^{s_1}_{ux}\rho^{s_2}_{vy}\rho^{s_2}_{wz}/\!/tm^{vw}_v/\!/hm^{xy}_x/\!/tha^{ux} = \rho^{s_2}_{vx}\rho^{s_2}_{wz}\rho^{s_1}_{uy}/\!/tm^{vw}_v/\!/hm^{xy}_x$.

Λ-calculus. $\Lambda(T;H) = R(T)\otimes(\Lambda(T)\otimes\Lambda(H))_=$, with $R(T)$ Laurent polynomials in $\{T_u\}_{u\in T}$. $\qquad \lambda_1*\lambda_2 = \lambda_1(\wedge\otimes\wedge)\lambda_2$

$tm^{uv}_w\colon\ u,v\to w,\ T_u,T_v\to T_w \qquad hm^{xy}_z\colon\ x\to z,\ y\to\sigma_xz \qquad tha^{ux}\colon\lambda\mapsto(1+i_u\otimes i_x)\lambda/\!/(u\to\sigma_xu) \qquad \rho^{\pm}_{ux} = 1+(T_u^{\pm1}-1)ux$

To do. • Full verification program. • Precise relation with Burau/Gassner. • Concordance. • Unitarity. • Planarity. • A depth-mirror property for u-objects. • Mutations? • Link relations? • Behaviour of A/MVA under mirror/strand reversal?