Pensieve Header: β-better calculus with explicit dm, continues bbCalculus.nb of pensieve://2012-05/.

```
bbSimplify = Simplify;
SetAttributes[bbCollect, Listable];
bbCollect[B[ω_, σ_, μ_]] := B[
   bbSimplify[ω], σ,
   Collect[μ, _h, Collect[#, _t, bbSimplify] &]
  ];
hL[b_] := Union[Cases[b, h[s_] :→ s, Infinity]];
tL[b_] := Union[Cases[b, t[s_] | T_s_ :→ s, Infinity]];
dL[b_] := Union[hL[b], tL[b]];
σ_ ⊢ h_ := (∂_h σ /. 0 → 1);
bbForm[B[ω_, σ_, μ_]] := Module[
   {tails, heads, mat},
   tails = tL[B[ω, σ, μ]]; heads = hL[B[ω, σ, μ]];
   mat = Outer[bbSimplify[∂_{h[#1],t[#2]} μ] &, heads, tails];
   PrependTo[mat, t /@ tails];
   mat = Join[
     {Prepend[h /@ heads, ω]},
     Transpose[mat],
     {Prepend[(σ ⊢ h[#]) & /@ heads, "1+Σ/ω"]}
    ];
   MatrixForm[mat]
  ];
bbForm[else_] := else /. b_B :→ bbForm[b];
Format[b_B, StandardForm] := bbForm[b];
B /: B[ω1_, σ1_, μ1_] == B[ω2_, σ2_, μ2_] := Module[
   {heads, tails},
   tails = tL[{B[ω1, σ1, μ1], B[ω2, σ2, μ2]}];
   heads = hL[{B[ω1, σ1, μ1], B[ω2, σ2, μ2]}];
   (ω1 == ω2) && (σ1 == σ2) && (
     And @@ Flatten[Outer[
        (Coefficient[μ1, t[#1] h[#2]] == Coefficient[μ2, t[#1] h[#2]]) &,
        tails, heads
       ]]
    )
  ];

B /: B[ω1_, σ1_, μ1_] B[ω2_, σ2_, μ2_] := B[ω1 * ω2, σ1 + σ2, ω2 μ1 + ω1 μ2];
tm[x_, y_, z_][b_] := b /. {t[x] → t[z], t[y] → t[z], T_x → T_z, T_y → T_z};
hm[x_, y_, z_][B[ω_, σ_, μ_]] := B[ω,
    h[z] (σ ⊢ h[x]) (σ ⊢ h[y]) + (σ /. h[x] | h[y] → 0),
    h[z] (D[μ, h[x]] + (σ ⊢ h[x]) ∂_{h[y]} μ) + (μ /. h[x] | h[y] → 0)
   ] // bbCollect;
swapth[y_, x_][B[ω_, σ_, μ_]] := Module[
   {α, β, γ, δ},
   (α β)   (Coefficient[μ, t[y] h[x]]  D[μ, t[y]] /. h[x] → 0)
   (γ δ) = (D[μ, h[x]] /. t[y] → 0     μ /. h[x] | t[y] → 0   );
```

```
    B[ω + α, σ,
      {(σ ⊢ h[x]) t[y], 1}.( α          β
                             γ  ((ω + α) δ - γ * β) / ω ).{h[x], 1}] // bbCollect
   ];
dm0[x_, y_, z_][b_] := b // swapth[x, y] // hm[x, y, z] // tm[x, y, z];
dm[x_, y_, z_][B[ω0_, σ_, μ_]] := Module[
   {ω, α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, σx, σy},
   ω = ω0 /. {T_x → T_z, T_y → T_z};
   {σx, σy} = {σ ⊢ h[x], σ ⊢ h[y]} /. {T_x → T_z, T_y → T_z};
   ( α  β  θ
     γ  δ  ϵ
     ϕ  ψ  Ξ ) =
    (        ∂_{t[x],h[x]} μ                  ∂_{t[x],h[y]} μ              ∂_{t[x]} μ /. h[x] | h[y] → 0
             ∂_{t[y],h[x]} μ                  ∂_{t[y],h[y]} μ              ∂_{t[y]} μ /. h[x] | h[y] → 0
     ∂_{h[x]} μ /. t[x] | t[y] → 0  ∂_{h[y]} μ /. t[x] | t[y] → 0  μ /. t[x] | t[y] | h[x] | h[y] →

    ) /. {T_x → T_z, T_y → T_z};
   B[ω + β,
     h[z] σx σy + (σ /. h[x] | h[y] → 0 /. {T_x → T_z, T_y → T_z}),
     {t[z], 1}.( σy α + γ + (β γ - δ α)/ω + σx σy β + σx δ    σy θ + ϵ + (β ϵ - δ θ)/ω
                       ϕ + (β ϕ - ψ α)/ω + σx ψ                  Ξ + (β Ξ - ψ θ)/ω      ).{h[z], 1}
    ] // bbCollect
  ]

Unprotect[NonCommutativeMultiply];
b1_B ** b2_B := Module[
   {ρ, σ, labels},
   ρ = b1 * (b2 /. {h[s_] ⧴ h[σ[s]], t[s_] ⧴ t[σ[s]], T_s_ ⧴ T_σ[s]});
   labels = dL[{b1, b2}];
   Do[ρ = ρ // dm[s, σ[s], s], {s, labels}];
   ρ
  ];

Rp[x_, y_] := B[1, T_x h[y], (T_x - 1) * t[x] h[y]];
Rm[x_, y_] := B[1, h[y] / T_x, (1 / T_x - 1) * t[x] h[y]];

{Rp[1, 2], Rm[1, 2]}
```

$$\left\{\begin{pmatrix} 1 & \mathrm{h}[2] \\ \mathrm{t}[1] & -1+T_1 \\ 1+\Sigma/\omega & T_1 \end{pmatrix}, \begin{pmatrix} 1 & \mathrm{h}[2] \\ \mathrm{t}[1] & -1+\frac{1}{T_1} \\ 1+\Sigma/\omega & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
Rp[1, 2] ** Rp[1, 3]
```

$$\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ 1+\Sigma/\omega & 1 & T_1 & T_1 \end{pmatrix}$$

```
{Rp[1, 2] ** Rp[1, 3] ** Rp[2, 3], Rp[2, 3] ** Rp[1, 3] ** Rp[1, 2]}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ t[2] & 0 & 0 & T_1(-1+T_2) \\ 1+\Sigma/\omega & 1 & T_1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & -1+T_1 & -1+T_1 \\ t[2] & 0 & 0 & T_1(-1+T_2) \\ 1+\Sigma/\omega & 1 & T_1 & T_1 T_2 \end{pmatrix}\right\}$$

```
<< KnotTheory`
{b = Times @@ (PD[Knot[8, 17]] /.
     X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], Rp[l, i], Rm[j, i]]);
 Do[b = dm[1, k, 1][b], {k, 2, 16}]; b,
 Alexander[Knot[8, 17]][T1] // bbSimplify
}
```

Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.

KnotTheory::loading : Loading precomputed data in PD4Knots`.

$$\left\{\begin{pmatrix} -8-\frac{1}{T_1^2}+\frac{4}{T_1}+11T_1-8T_1^2+4T_1^3-T_1^4 & h[1] \\ t[1] & 0 \\ 1+\Sigma/\omega & 1 \end{pmatrix}, 11-\frac{1}{T_1^3}+\frac{4}{T_1^2}-\frac{8}{T_1}-8T_1+4T_1^2-T_1^3\right\}$$

$$\begin{pmatrix} \sigma y\,\alpha+\gamma+\frac{\beta\gamma-\delta\alpha}{\omega}+\sigma x\,\sigma y\,\beta+\sigma x\,\delta & \sigma y\,\theta+\epsilon+\frac{\beta\epsilon-\delta\theta}{\omega} \\ \phi+\frac{\beta\phi-\psi\alpha}{\omega}+\sigma x\,\psi & \Xi+\frac{\beta\Xi-\psi\theta}{\omega} \end{pmatrix} \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} \gamma+\delta\,\sigma x+\alpha\,\sigma y+\beta\,\sigma x\,\sigma y+\frac{\beta\gamma-\alpha\delta}{\omega} & \epsilon+\theta\,\sigma y+\frac{\beta\epsilon-\delta\theta}{\omega} \\ \phi+\sigma x\,\psi+\frac{\beta\phi-\alpha\psi}{\omega} & \Xi+\frac{\beta\Xi-\theta\psi}{\omega} \end{pmatrix}$$

$$\text{B0 = B}\left[\omega,\ \{\sigma_a, \sigma_b, \sigma\}.\{h@a, h@b, h@S\}, \{t@a, t@b, t@S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h@a, h@b, h@S\}\right]$$

$$\begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \alpha & \beta & \theta \\ t[b] & \gamma & \delta & \epsilon \\ t[S] & \phi & \psi & \Xi \\ 1+\Sigma/\omega & \sigma_a & \sigma_b & \sigma \end{pmatrix}$$

```
B0 // swapth[a, b] // hm[a, b, c] // tm[a, b, c]
```

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \sigma_a(\delta+\beta\sigma_b)+\frac{\beta\gamma-\alpha\delta+\gamma\omega+\alpha\omega\sigma_b}{\omega} & \frac{\beta\epsilon-\delta\theta+\epsilon\omega+\theta\omega\sigma_b}{\omega} \\ t[S] & \frac{\beta\phi-\alpha\psi+\phi\omega+\psi\omega\sigma_a}{\omega} & \frac{\beta\Xi-\theta\psi+\Xi\omega}{\omega} \\ 1+\Sigma/\omega & \sigma_a\sigma_b & \sigma \end{pmatrix}$$

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \sigma_a\,(\delta+\beta\,\sigma_b)+\frac{\beta\,\gamma-\alpha\,\delta+\gamma\,\omega+\alpha\,\omega\,\sigma_b}{\omega} & \frac{\beta\,\epsilon-\delta\,\Theta+\epsilon\,\omega+\Theta\,\omega\,\sigma_b}{\omega} \\ t[S] & \frac{\beta\,\phi-\alpha\,\psi+\phi\,\omega+\psi\,\omega\,\sigma_a}{\omega} & \frac{\beta\,\Xi-\Theta\,\psi+\Xi\,\omega}{\omega} \\ \text{"}1+\Sigma/\omega\text{"} & \sigma_a\,\sigma_b & \sigma \end{pmatrix}$$ // FullSimplify // MatrixForm

$$\begin{pmatrix} \beta+\omega & h[c] & h[S] \\ t[c] & \frac{-\alpha\,\delta+\gamma\,(\beta+\omega)+\alpha\,\omega\,\sigma_b+\omega\,\sigma_a\,(\delta+\beta\,\sigma_b)}{\omega} & \frac{-\delta\,\Theta+\epsilon\,(\beta+\omega)+\Theta\,\omega\,\sigma_b}{\omega} \\ t[S] & \frac{-\alpha\,\psi+\phi\,(\beta+\omega)+\psi\,\omega\,\sigma_a}{\omega} & \Xi+\frac{\beta\,\Xi-\Theta\,\psi}{\omega} \\ 1+\Sigma/\omega & \sigma_a\,\sigma_b & \sigma \end{pmatrix}$$