Cheat Sheet β

http://drorbn.net/AcademicPensieve/2014-04/; verification at 2014-04/CheatSheetBeta-Verification.nb
initiated 24/3/13; continues 2013-06; continued 2014-05; modified 1/5/14, 10:02am

β-calculus. With $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$,

$$\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array} \underset{\beta}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array} \qquad \begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \gamma\end{array} \xrightarrow[\beta]{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \gamma\end{array}\right) /\!\!/ \left(\begin{smallmatrix}u,v\\ \to w\end{smallmatrix}\right) \qquad \rho^\pm_{ux} \underset{\beta}{=} \begin{array}{c|c}1 & x\\ \hline u & t_u^{\pm1}-1\end{array}$$

$$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \gamma\end{array} \xrightarrow[\beta]{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\beta+\langle\alpha\rangle\beta & \gamma\end{array} \qquad \begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \beta\\ T & \gamma & \delta\end{array} \xrightarrow[\beta]{sw^{ux}_{th}} \begin{array}{c|cc}\omega\epsilon & x & H\\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon)\\ T & \gamma/\epsilon & \delta-\gamma\beta/\epsilon\end{array}$$

Constraints. • Column sums are monomials minus 1. • At $t_*=1$, $\omega=1$ and $A=0$.

σ calculus. $\sigma_1*\sigma_2=\sigma_1\cup\sigma_2$, $tm^{uv}_w = \left(\begin{smallmatrix}u,v\\ \to w\end{smallmatrix}\right)$, $\sigma \xrightarrow{hm^{xy}_z} (\sigma\setminus\{x,y\})\cup(z\to\sigma_x\sigma_y)$, $sw^{ux}_{th}=I$, $R^\pm_{ux}\to t_u^{\pm1}$

β-better calculus. Constraints. • Sum of column x is $(\sigma_x-1)w$. • $\omega^{k-1}\mid\Lambda^k A$. • At $t_*=1$, $\omega=1$ and $A=0$.

$$\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array} \underset{\beta_b}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & \omega_2A_1 & 0\\ T_2 & 0 & \omega_1A_2\end{array} \qquad \begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \gamma\end{array} \xrightarrow[\beta_b]{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \gamma\end{array}\right) /\!\!/ \left(\begin{smallmatrix}u,v\\ \to w\end{smallmatrix}\right) \qquad \rho^\pm_{ux} \underset{\beta_b}{=} \begin{array}{c|c}1 & x\\ \hline u & t_u^{\pm1}-1\end{array}$$

$$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \gamma\end{array} \xrightarrow[\beta_b]{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \gamma\end{array} \qquad \begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \beta\\ T & \gamma & \delta\end{array} \xrightarrow[\beta_b]{sw^{ux}_{th}} \begin{array}{c|cc}\omega+\alpha & x & H\\ \hline u & \sigma_x\alpha & \sigma_x\beta\\ T & \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{array} =: \begin{array}{c|c}\cdot & -\\ \hline | & \begin{pmatrix}\sigma_x & 0\\ 0 & 1\end{pmatrix}\cdot A^{ux}\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\beta_b\ \checkmark]{dm^{ab}_c} \left(\begin{array}{c|cc}\omega+\beta & c & S\\ \hline c & \gamma+\sigma_a\delta+\sigma_b(\alpha+\sigma_a\beta)+\frac{\beta\gamma-\alpha\delta}{\omega} & \epsilon+\sigma_b\theta+\frac{\beta\epsilon-\delta\theta}{\omega}\\ S & \phi+\sigma_a\psi+\frac{\beta\phi-\alpha\psi}{\omega} & \Xi+\frac{\beta\Xi-\psi\theta}{\omega}\end{array}\right) /\!\!/ \left(\begin{smallmatrix}t_a,t_b\\ \to t_c\end{smallmatrix}\right)$$

The MVA (mod units): n-component $L \underset{\beta b}{\mapsto} (\sigma,\omega,A) \mapsto \dfrac{\omega^{2-n}\det'(A-\omega\operatorname{diag}((\sigma_i-1))}{1-t_1}$ ✓

Note. $A^{ux} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \frac{(\omega+\alpha)\delta-\gamma\beta}{\omega}\end{pmatrix} = \frac{1}{\omega}\left[(\omega+\alpha)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} - \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix}\right] = \frac{1}{\omega}\left[(\omega+a_{ux})A - a_{*x}a_{u*}\right].$

Claim. $\omega^{k-1}\mid\Lambda^kA$ and $\omega^k\mid\Lambda^{k+1}A$ implies $(\omega+\alpha)^{k-1}\mid\Lambda^kA^{ux}$, with $\alpha=a_{ux}$.

Proof. With $\bar u\in T^k$ and $\bar x\in H^k$, ω^k divides $\begin{vmatrix}\omega & 0\\ 0 & a_{\bar u\bar x}\end{vmatrix}$ and $\begin{vmatrix}a_{ux} & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix}$ and hence their sum, $\begin{vmatrix}\omega+\alpha & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix} = (\omega+\alpha)\begin{vmatrix}1 & 0\\ 0 & a_{\bar u\bar x}-\frac{1}{\omega+\alpha}a_{\bar ux}a_{u\bar x}\end{vmatrix} = \frac{1}{(\omega+\alpha)^{k-1}}\left|(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right|$. So $\frac{1}{(\omega+\alpha)^{k-1}}\left|\frac{1}{\omega}\left[(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right]\right|$ is integral. □

That is, with $A_{\bar u;\bar x}$ denoting minors, if $\omega^{k-1}\mu_{\bar u;\bar x}=A_{\bar u;\bar x}$ and $\omega^k\mu_{u\bar u;x\bar x}=A_{u\bar u;x\bar x}$, then $(\omega+\alpha)^{k-1}(\mu_{\bar u;\bar x}+\mu_{u\bar u;x\bar x})=A^{ux}_{\bar u;\bar x}$.

Relations. • $\rho^+_{ux}\rho^-_{vy} /\!\!/ tm^{uv}_w /\!\!/ hm^{xy}_z = t\epsilon_w h\epsilon_z$. • $\rho^{s_1}_{ux}\rho^{s_2}_{vy}\rho^{s_2}_{wz} /\!\!/ tm^{vw}_v /\!\!/ hm^{xy}_x /\!\!/ sw^{ux}_{th} = \rho^{s_2}_{vx}\rho^{s_2}_{wz}\rho^{s_1}_{uy} /\!\!/ tm^{vw}_v /\!\!/ hm^{xy}_x$.

Λ-calculus. $\Lambda(T;H)=R(T)\otimes(\Lambda(T)\otimes\Lambda(H))_=$, with $R(T)$ Laurent polynomials in $\{t_u\}_{u\in T}$. $\lambda_1*\lambda_2=\lambda_1(\wedge\otimes\wedge)\lambda_2$

$tm^{uv}_w:\ u,v\to w,\ t_u,t_v\to t_w$ $\quad hm^{xy}_z:\ x\to z,\ y\to\sigma_x z$ $\quad sw^{ux}_{th}:\lambda\mapsto(1+i_u\otimes i_x)\lambda /\!\!/ (u\to\sigma_x u)$ $\quad \rho^\pm_{ux}=1+(t_u^{\pm1}-1)ux$

To do. • A verification program. • Add Burau calculus.