

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a **semi-rigorous** reduction of perturbative BF theory (Cattaneo-Rossi [CR]) to computable combinatorics, in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. **Weak** this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. (good news in **highlight**)

The BF Feynman Rules. For an edge e , let Φ_e be its direction, in S^3 or S^1 . Let ω_3 and ω_1 be volume forms on S^3 and S^1 . Then for a 2-link $(\kappa_l)_{l \in T}$,



$$\zeta = \log \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{\text{red } e \in D} \Phi_e^* \omega_3 \prod_{\text{black } e \in D} \Phi_e^* \omega_1$$

is an invariant in $CW(FL(T)) \rightarrow CW(T)/\sim$, "symmetrized cyclic words in T ".

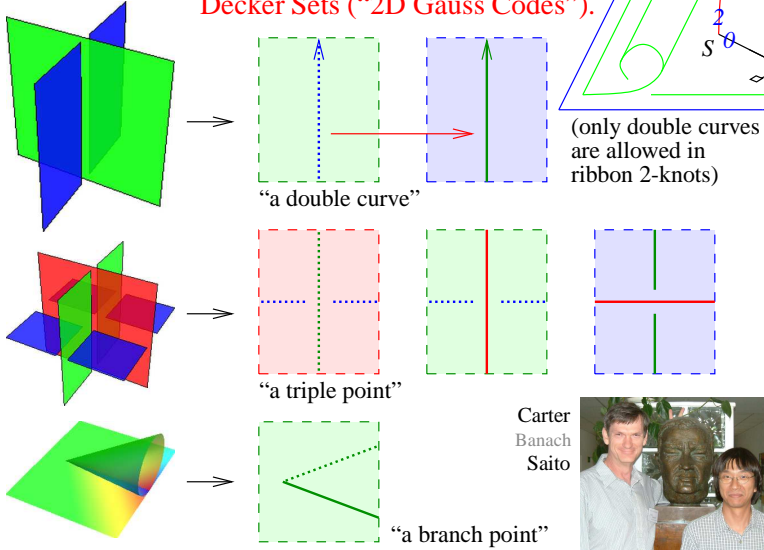
BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, g)$, $B \in \Omega^2(M, g^*)$,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

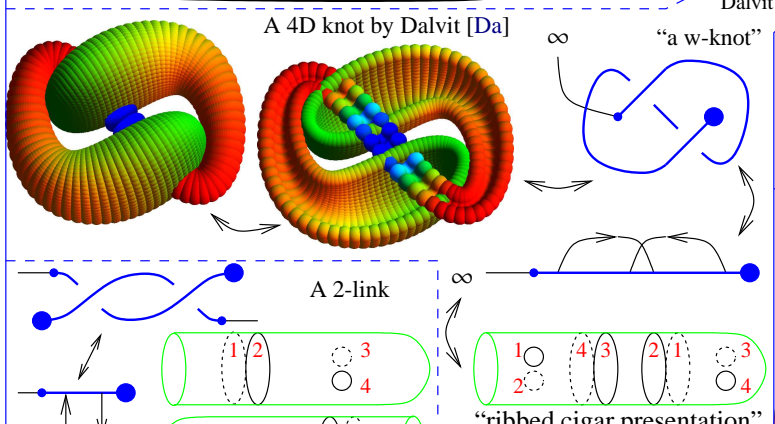
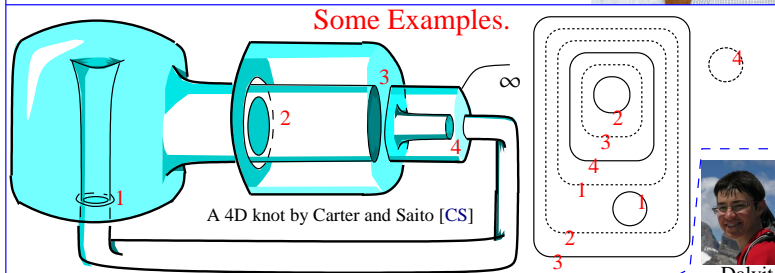
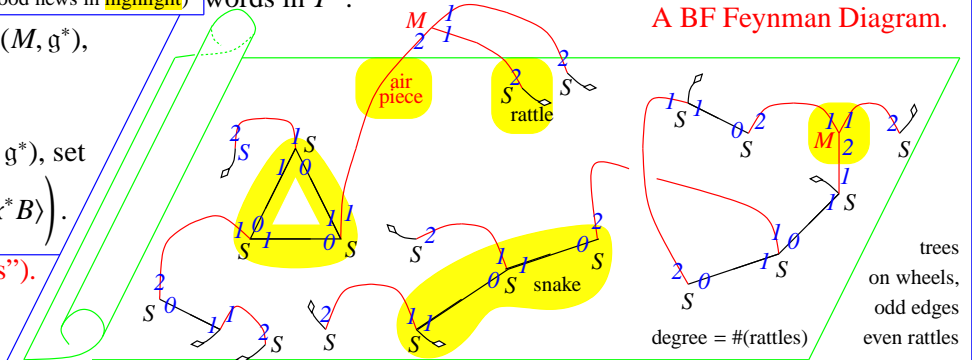
With $\kappa: (S = \mathbb{R}^2) \rightarrow M$, $\beta \in \Omega^0(S, g)$, $\alpha \in \Omega^1(S, g^*)$, set

$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d\kappa^* A \alpha + \kappa^* B \rangle\right).$$

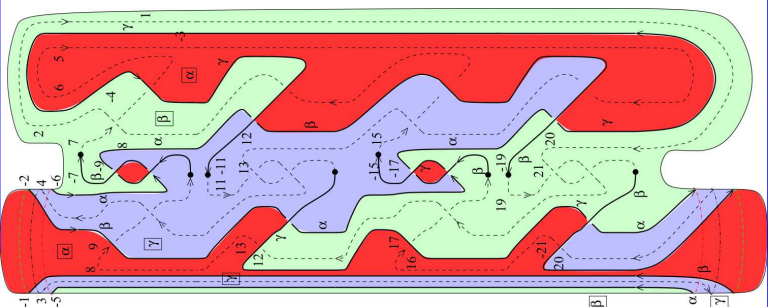
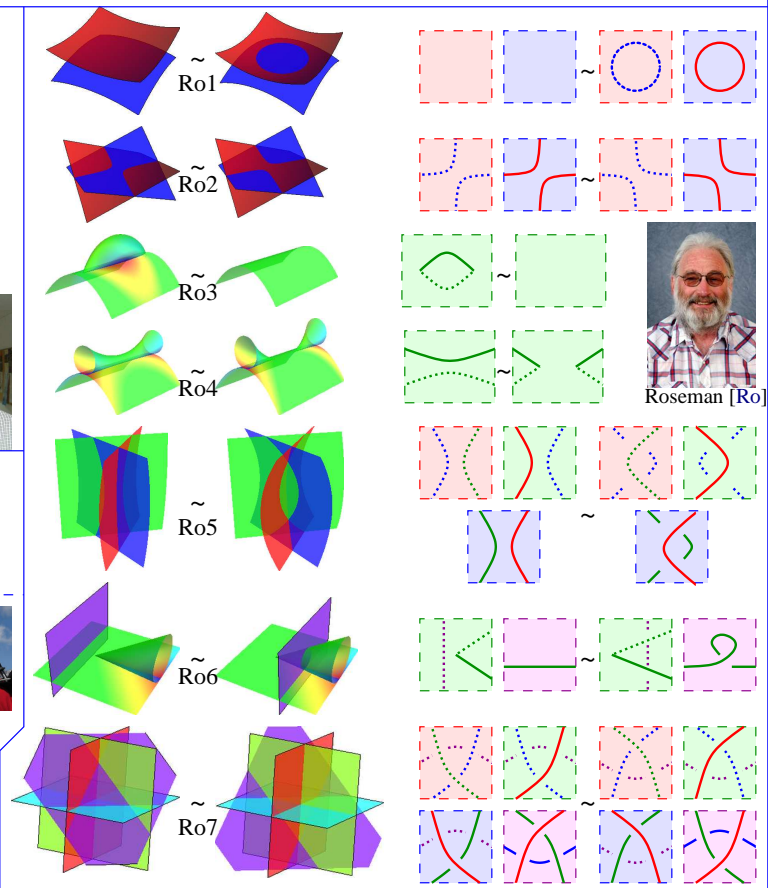
Decker Sets ("2D Gauss Codes").



A BF Feynman Diagram.



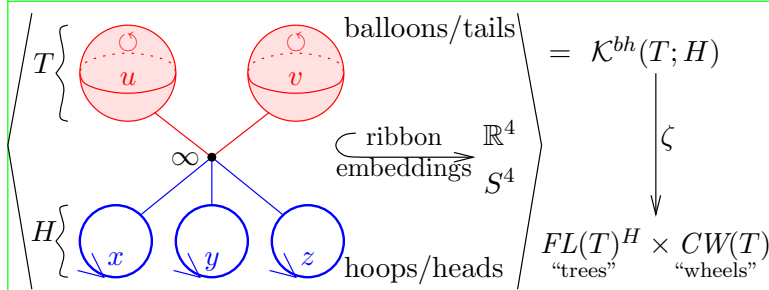
A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].



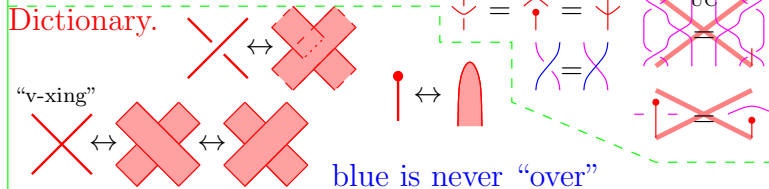
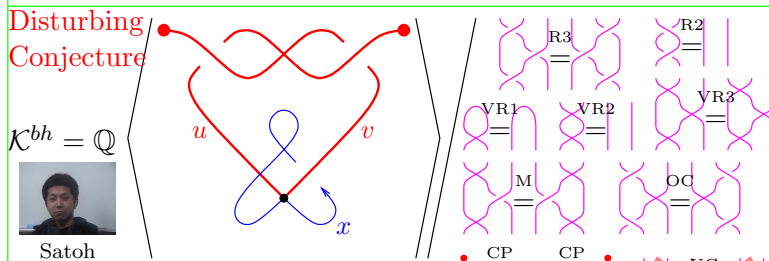


Finite Type Invariants of Ribbon Knotted Balloons and Hoops

Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.



My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.



Expansions
the semi-virtual $\otimes := \otimes - \otimes$ i.e. $\otimes - \otimes$ or $\otimes - \otimes$

Let $\mathcal{I}^n := \langle \text{pictures with } \geq n \text{ semi-virts} \rangle \subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z: \mathcal{K}^{bh} \rightarrow \text{gr } \mathcal{K}^{bh} = \widehat{\bigoplus} \mathcal{I}^n / \mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

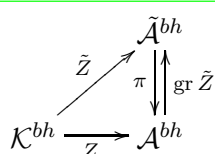
satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

$$Z(\gamma) = (0, \dots, 0, \gamma / \mathcal{I}^{n+1}, *, *, \dots).$$



Why? • Just because, and this is vastly more general.
• $(\mathcal{K}^{bh} / \mathcal{I}^{n+1})^*$ is "finite-type/polynomial invariants".
• The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K} : f(0) = 0\}$. Then $\mathcal{I}^n = \{f : f \text{ vanishes like } |x|^n\}$ so $\mathcal{I}^n / \mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi: \tilde{\mathcal{A}}^{bh} \rightarrow \mathcal{A}^{bh}$, and a filtered $\tilde{Z}: \mathcal{K}^{bh} \rightarrow \tilde{\mathcal{A}}^{bh}$ so that $\pi \circ \text{gr } \tilde{Z} = \text{Id}$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} \circ \pi$ is an expansion.



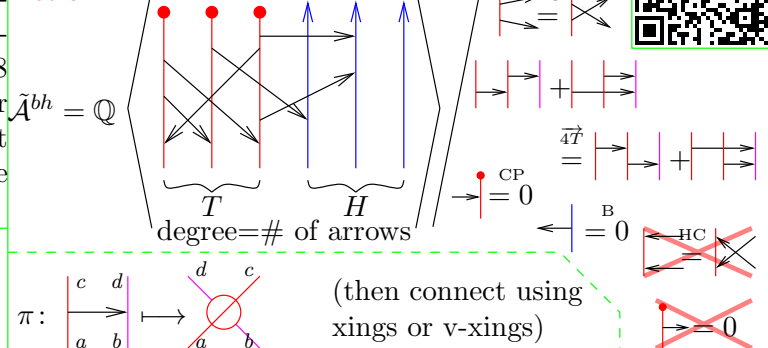
"God created the knots, all else in topology is the work of mortals."



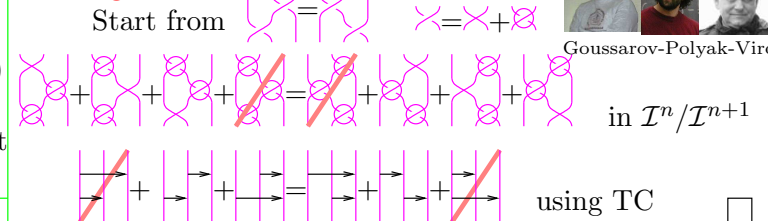
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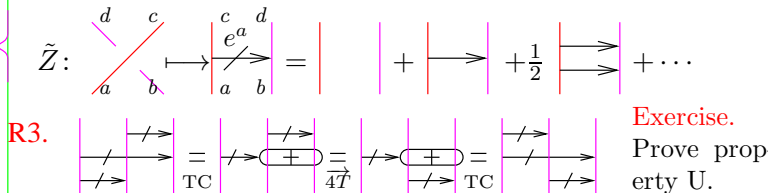
Action 1.



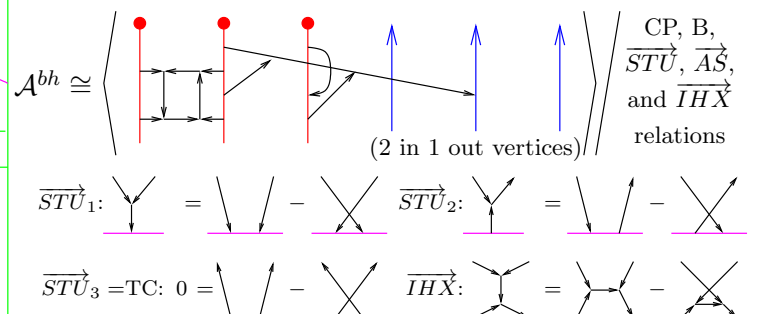
Deriving 4T.



Action 2.



The Bracket-Rise Theorem.



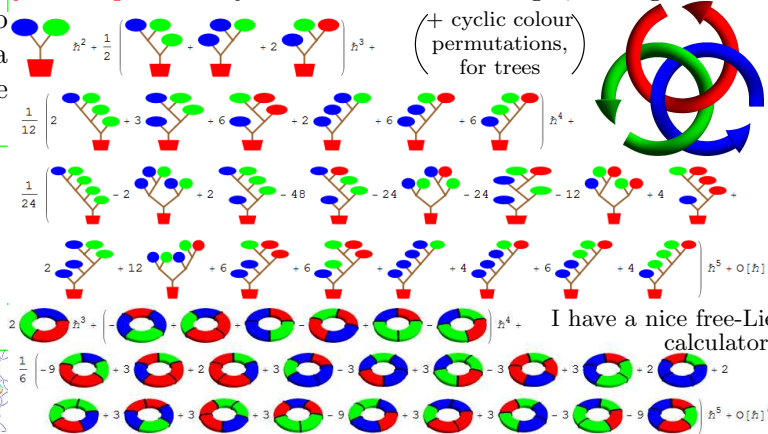
Proof.



Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. \mathcal{A}^{bh} is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Zurich, September 2013

$\omega\epsilon\beta$: <http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919>



15 Minutes on Algebra

Let T be a finite set of “tail labels” and H a finite set of “head labels”. Set

$$M_{1/2}(T; H) := FL(T)^H,$$

“ H -labeled lists of elements of the degree-completed free Lie algebra generated by T ”.

$$FL(T) = \left\{ 2t_2 - \frac{1}{2}[t_1, [t_1, t_2]] + \dots \right\} / \left(\begin{array}{c} \text{anti-symmetry} \\ \text{Jacobi} \end{array} \right)$$

... with the obvious bracket.

$$M_{1/2}(u, v; x, y) = \left\{ \lambda = \left(x \rightarrow \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ x \end{array}, y \rightarrow \begin{array}{c} v \\ \downarrow \\ y \end{array} - \frac{22}{7} \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ y \end{array} \right) \dots \right\}$$

Operations $M_{1/2} \rightarrow M_{1/2}$. ↙ newspeak!

Tail Multiply tm_w^{uv} is $\lambda \mapsto \lambda \parallel (u, v \rightarrow w)$, satisfies “meta-associativity”, $tm_u^{uv} \parallel tm_u^{vw} = tm_v^{uv} \parallel tm_u^{vw}$.

Head Multiply hm_z^{xy} is $\lambda \mapsto (\lambda \setminus \{x, y\}) \cup (z \rightarrow \text{bch}(\lambda_x, \lambda_y))$, where

$$\text{bch}(\alpha, \beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha, \beta]}{2} + \frac{[\alpha, [\alpha, \beta]] + [[\alpha, \beta], \beta]}{12} + \dots$$

satisfies $\text{bch}(\text{bch}(\alpha, \beta), \gamma) = \log(e^{\alpha} e^{\beta} e^{\gamma}) = \text{bch}(\alpha, \text{bch}(\beta, \gamma))$ and hence meta-associativity, $hm_x^{xy} \parallel hm_x^{yz} = hm_y^{xy} \parallel hm_x^{yz}$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda \parallel RC_u^{\lambda_x}$, where $C_u^{-\gamma}: FL \rightarrow FL$ is the substitution $u \rightarrow e^{-\gamma} u e^{\gamma}$, or more precisely,

$$C_u^{-\gamma}: u \rightarrow e^{-\text{ad } \gamma}(u) = u - [\gamma, u] + \frac{1}{2}[\gamma, [\gamma, u]] - \dots,$$

and $RC_u^{\gamma} = (C_u^{-\gamma})^{-1}$. Then $C_u^{\text{bch}(\alpha, \beta)} = C_u^{\alpha} \parallel RC_u^{-\beta} \parallel C_u^{\beta}$ hence $RC_u^{\text{bch}(\alpha, \beta)} = RC_u^{\alpha} \parallel RC_u^{\beta} \parallel RC_u^{\alpha}$ hence “meta $u^{xy} = (u^x)^y$ ”,

$$hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy},$$

and $tm_w^{uv} \parallel C_w^{\gamma} \parallel tm_w^{uv} = C_u^{\gamma} \parallel RC_u^{-\gamma} \parallel C_u^{\gamma} \parallel tm_w^{uv}$ and hence “meta $(uv)^x = u^x v^x$ ”, $tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}$.

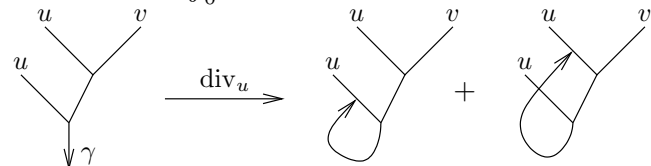
Wheels. Let $M(T; H) := M_{1/2}(T; H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T , or equally well, on $FL(T)$:



Operations. On $M(T; H)$, define tm_w^{uv} and hm_z^{xy} as before, and tha^{ux} by adding some J -spice:

$$(\lambda; \omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) \parallel RC_u^{\lambda_x},$$

where $J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}$, and

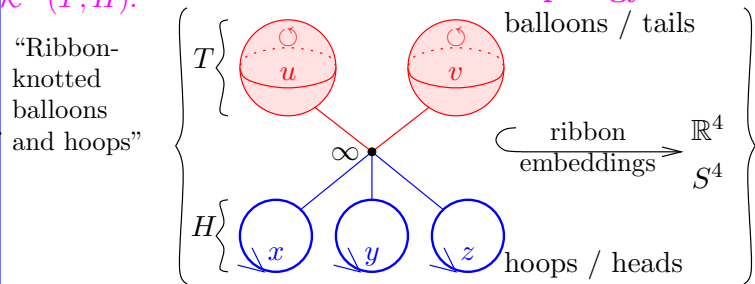


Theorem Blue. All blue identities still hold.

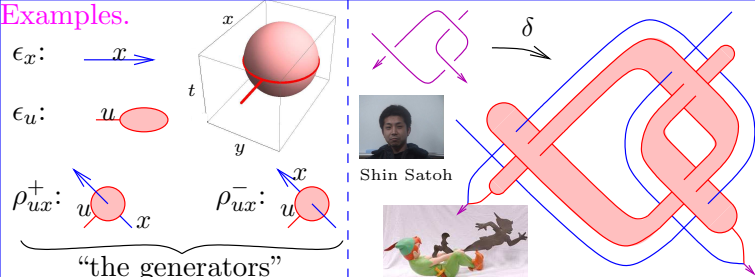
Merge Operation. $(\lambda_1; \omega_1) * (\lambda_2; \omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

$\mathcal{K}^{bh}(T; H)$.

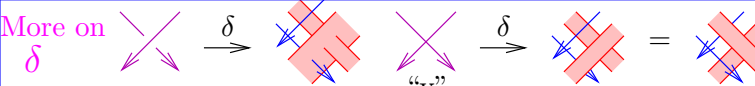
15 Minutes on Topology



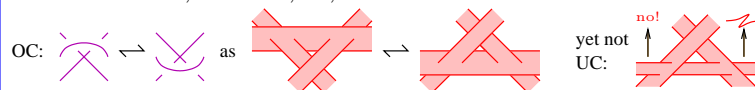
Examples.



More on δ

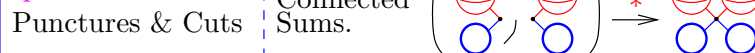


satisfies R123, VR123, D, and



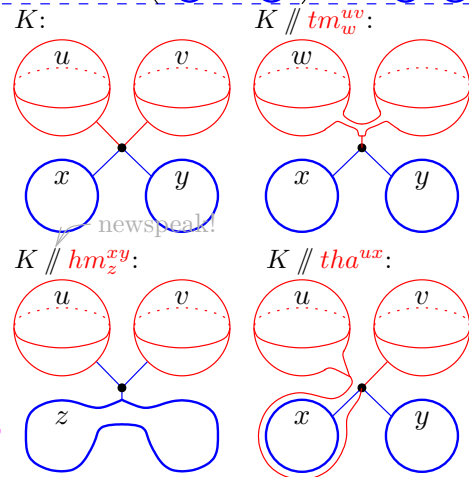
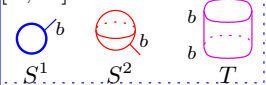
- δ injects u-knots into \mathcal{K}^{bh} (likely u-tangles too).
- δ maps v-tangles to \mathcal{K}^{bh} ; the kernel contains the above and **conjecturally** (Satoh), that’s all.
- Allowing punctures and cuts, δ is onto.

Operations



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2 .

Riddle. People often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$. Why not $\pi_T(X) := [T, X]$?

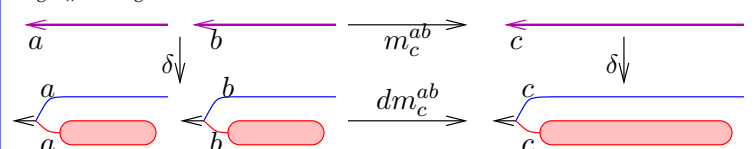


“Meta-Group-Action”

Properties.

- Associativities: $m_a^{ab} \parallel m_a^{ac} = m_b^{bc} \parallel m_a^{ab}$, for $m = tm, hm$.
- “(uv)^x = u^xv^x”: $tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}$,
- “(u(xy) = (u^x)^y”: $hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy}$.

Tangle concatenations $\rightarrow \pi_1 \times \pi_2$. With $dm_c^{ab} := tha^{ab} \parallel tm_c^{ab} \parallel hm_c^{ab}$,



Finite type invariants make sense in the usual way, and “algebra” is (the primitive part of) “gr” of “topology”.

Trees and Wheels and Balloons and Hoops: Why I Care

Moral. To construct an M -valued invariant ζ of (v-)tangles, and nearly an invariant on \mathcal{K}^{bh} , it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

The Invariant ζ . Set $\zeta(\epsilon_x) = (x \rightarrow 0; 0)$, $\zeta(\epsilon_u) = ((); 0)$, and

$$\zeta: \begin{array}{c} \text{diagram of } \epsilon_x \\ \text{diagram of } \epsilon_u \end{array} \mapsto \begin{array}{c} \left(\begin{array}{c} u \\ \downarrow x \\ x \end{array}; 0 \right) \\ \left(- \begin{array}{c} u \\ \downarrow x \\ x \end{array}; 0 \right) \end{array}$$

Theorem. ζ is (log of) the unique homomorphic universal finite type invariant on \mathcal{K}^{bh} .
 (... and is the tip of an iceberg)

Paper in progress with Danco, $\omega\epsilon\beta/wko$



See also $\omega\epsilon\beta/tenn$, $\omega\epsilon\beta/bonn$, $\omega\epsilon\beta/swiss$, $\omega\epsilon\beta/portfolio$

ζ is computable! ζ of the Borromean tangle, to degree 5:

I have a nice free-Lie calculator!

Tensorial Interpretation. Let \mathfrak{g} be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \rightarrow \text{Fun}(\oplus T\mathfrak{g} \rightarrow \mathfrak{g})$ and $\tau : CW(T) \rightarrow \text{Fun}(\oplus T\mathfrak{g})$. Together, $\tau : M(T; H) \rightarrow \text{Fun}(\oplus T\mathfrak{g} \rightarrow \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T; H) \rightarrow \text{Fun}(\oplus T\mathfrak{g} \rightarrow \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a \mathfrak{g} -connection on S^4 with curvature F_A , and B a \mathfrak{g}^* -valued 2-form on S^4 . For a hoop γ_x , let $\text{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x . For a ball γ_u , let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be (roughly) the integral of B (transported via A to ∞) on γ_u .



Loose Conjecture. For $\gamma \in \mathcal{K}(T; H)$,

$$\int \mathcal{D}A \mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \bigotimes_x \text{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

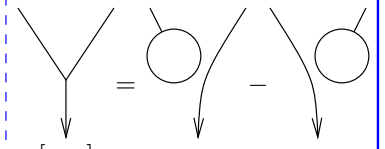
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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The β quotient is M divided by all relations that universally hold when \mathfrak{g} is the 2D non-Abelian Lie algebra. Let $R = \mathbb{Q}[\{c_u\}_{u \in T}]$ and $L_\beta := R \otimes T$ with central R and with $[u, v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \rightarrow L_\beta$ and $CW \rightarrow R$. Under this,



$$\mu \rightarrow ((\lambda_x); \omega) \quad \text{with } \lambda_x = \sum_{u \in T} \lambda_{ux} u x, \quad \lambda_{ux}, \omega \in R,$$

$$\text{bch}(u, v) \rightarrow \frac{c_u + c_v}{e^{c_u + c_v} - 1} \left(\frac{e^{c_u} - 1}{c_u} u + e^{c_u} \frac{e^{c_v} - 1}{c_v} v \right),$$

if $\gamma = \sum \gamma_v v$ then with $c_\gamma := \sum \gamma_v c_v$,

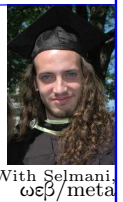
$$u // RC_\gamma^u = \left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \left(e^{c_\gamma} u - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right),$$

$\text{div}_u \gamma = c_u \gamma_u$, and $J_u(\gamma) = \log \left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u \right)$, so ζ is formula-computable to all orders! **Can we simplify?**

Repackaging. Given $((x \rightarrow \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \rightarrow \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x} - 1}{c_x}$ and $\omega \rightarrow e^\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get " **β calculus**".

β Calculus. Let $\beta(T; H)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} \omega \text{ and the } \alpha_{ux}'\text{s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T. \end{array} \right\},$$



$$tm_w^{uv} : \begin{array}{c|c} \omega & \cdots \\ u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ w & \alpha + \beta \\ \vdots & \gamma \end{array}, \quad \begin{array}{c|cc} \omega_1 & H_1 & \omega_2 & H_2 \\ T_1 & \alpha_1 & T_2 & \alpha_2 \\ \hline & \omega_1 \omega_2 & H_1 & H_2 \\ & T_1 & \alpha_1 & 0 \\ & T_2 & 0 & \alpha_2 \end{array},$$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & z \\ \vdots & \alpha + \beta + \langle \alpha \rangle \beta \\ \vdots & \gamma \end{array},$$

$$tha_{ux} : \begin{array}{c|ccc} \omega & x & \cdots \\ u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega \epsilon & x & \cdots \\ u & \alpha(1 + \langle \gamma \rangle / \epsilon) & \beta(1 + \langle \gamma \rangle / \epsilon) \\ \vdots & \gamma / \epsilon & \delta - \gamma \beta / \epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle \alpha \rangle := \sum_v \alpha_v$, and $\langle \gamma \rangle := \sum_{v \neq u} \gamma_v$, and let

$$R_{ux}^+ := \frac{1}{u} \left| \begin{array}{c} x \\ t_u - 1 \end{array} \right. \quad R_{ux}^- := \frac{1}{u} \left| \begin{array}{c} x \\ t_u^{-1} - 1 \end{array} \right.$$

On long knots, ω is the Alexander polynomial!

Why happy? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*



May class: $\omega\epsilon\beta/aarhus$

Class next year: $\omega\epsilon\beta/1350$

Paper: $\omega\epsilon\beta/kbh$

Meta-Groups, Meta-Bicrossed-Products, and the Alexander Polynomial, 1



Dror Bar-Natan in Montreal, June 2013.

<http://www.math.toronto.edu/~drorbn/Talks/Montreal-1306/>

Abstract. I will define “meta-groups” and explain how one specific meta-group, which in itself is a “meta-bicrossed-product”, gives rise to an “ultimate Alexander invariant” of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that’s a wonderful playground.

This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).

See also Dror Bar-Natan and Sam Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, arXiv:1302.5689.

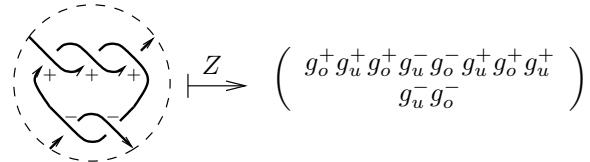
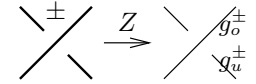


Sam Selmani

Alexander Issues.

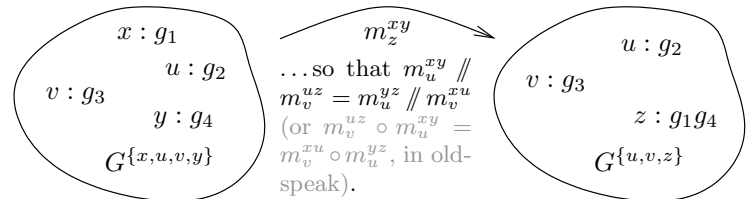
- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

Idea. Given a group G and two “YB” pairs $R^\pm = (g_o^\pm, g_u^\pm) \in G^2$, map them to xings and “multiply along”, so that



This Fails! R2 implies that $g_o^\pm g_o^\mp = e = g_u^\pm g_u^\mp$ and then R3 implies that g_o^\pm and g_u^\pm commute, so the result is a simple counting invariant.

A Group Computer. Given G , can store group elements and perform operations on them:



Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ_{xy}^z for element cloning, ρ_y^x for renamings, and $(D_1, D_2) \mapsto D_1 \cup D_2$ for merging, and many obvious composition axioms relating those.

$$P = \{x : g_1, y : g_2\} \Rightarrow P = \{d_y P\} \cup \{d_x P\}$$

A Meta-Group. Is a similar “computer”, only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ , and a collection of operations m_z^{xy} , S_x , e_x , d_x , Δ_{xy}^z (sometimes), ρ_y^x , and \cup , satisfying the exact same *linear* properties.

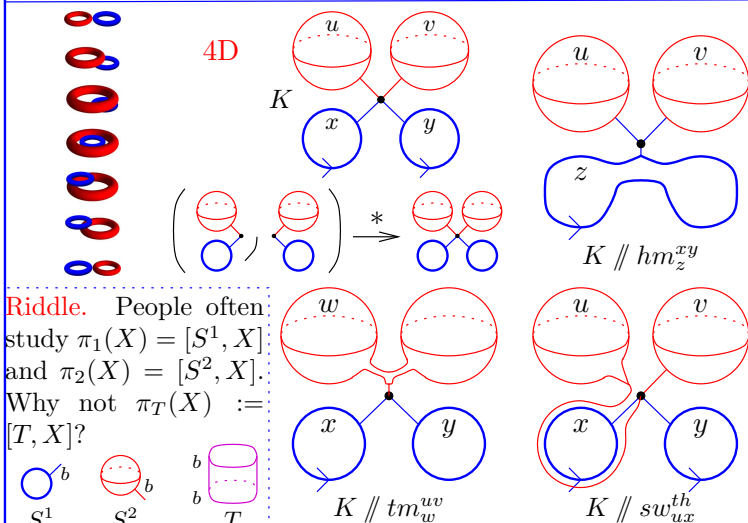
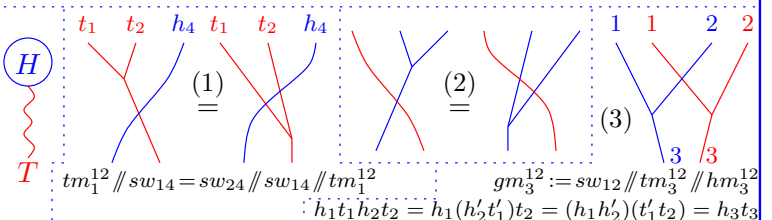
Example 0. The non-meta example, $G_\gamma := G^\gamma$.

Example 1. $G_\gamma := M_{\gamma \times \gamma}(\mathbb{Z})$, with simultaneous row and column operations, and “block diagonal” merges. Here if $P = \begin{pmatrix} x & a & b \\ y & c & d \end{pmatrix}$ then $d_y P = (x : a)$ and $d_x P = (y : d)$ so

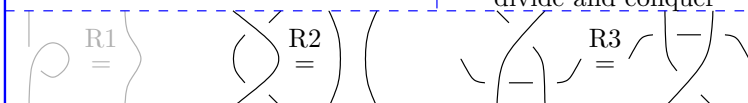
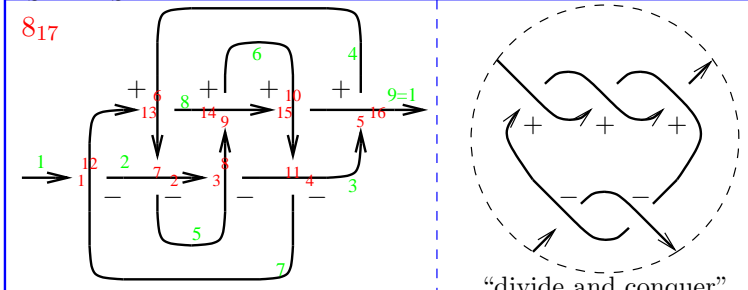
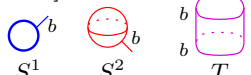
$$\{d_y P\} \cup \{d_x P\} = \begin{pmatrix} x & a & 0 \\ y & 0 & d \end{pmatrix} \neq P. \text{ So this } G \text{ is truly meta.}$$

Claim. From a meta-group G and YB elements $R^\pm \in G_2$ we can construct a knot/tangle invariant.

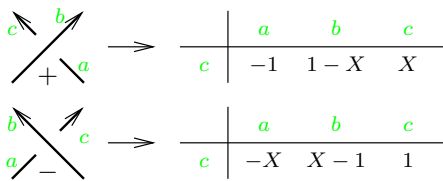
Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{e\}$, then also $G = TH$ and G is determined by H , T , and the “swap” map $sw^{th} : (t, h) \mapsto (h', t')$ defined by $th = h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw : T \times H \rightarrow H \times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H \times T$, the “bicrossed product”.



Riddle. People often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$. Why not $\pi_T(X) := [T, X]$?



A Standard Alexander Formula. Label the arcs 1 through $(n + 1) = 1$, make an $n \times n$ matrix as below, delete one row and one column, and compute the determinant:



$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & x-1 & 0 & -x \\ -1 & x & 0 & 0 & 0 & 0 & 0 & 1-x & 0 \\ 0 & -1 & x & 0 & 1-x & 0 & 0 & 0 & 0 \\ x-1 & 0 & -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-x & 0 & -1 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 1 & 0 & 0 & x-1 \\ 0 & 0 & 1-x & 0 & 0 & -1 & x & 0 & 0 \\ 0 & 0 & 0 & x-1 & 0 & 0 & -x & 1 & 0 \end{pmatrix} \quad [[1 ;; 7, 1 ;; 7]] // \text{Det}$$

$$-1 + 4x - 8x^2 + 11x^3 - 8x^4 + 4x^5 - x^6$$

$$h_1 t_1 h_2 t_2 = h_1 (h_2' t_1') t_2 = (h_1 h_2') (t_1' t_2) = h_3 t_3$$

Meta-Groups, Meta-Bicrossed-Products, and the Alexander Polynomial, 2

A **Meta-Bicrossed-Product** is a collection of sets $\beta(\eta, \tau)$ and operations tm_w^{uv} , hm_z^{xy} and sw_{ux}^{th} (and lesser ones), such that tm and hm are “associative” and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma, \gamma)$ and gm as in (3).

Example. Take $\beta(\eta, \tau) = M_{\tau \times \eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta, \tau)$ be

$$\left\{ \begin{array}{c|ccc} \omega & h_1 & h_2 & \cdots \\ t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} h_j \in \eta, t_i \in \tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in a variable } X \text{ with} \\ \omega(1) = 1 \text{ and } \alpha_{ij}(1) = 0 \end{array} \right\},$$

$$tm_w^{uv} : \begin{array}{c|c} \omega & \cdots \\ t_u & \alpha \\ t_v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ t_w & \alpha + \beta \\ \vdots & \gamma \end{array}, \quad \begin{array}{c|c} \omega_1 & \eta_1 \\ \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \tau_2 & \alpha_2 \end{array} = \begin{array}{c|c} \omega_1\omega_2 & \eta_1 \eta_2 \\ \tau_1 & \alpha_1 \ 0 \\ \tau_2 & 0 \ \alpha_2 \end{array},$$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|ccc} \omega & h_z & & \cdots \\ \vdots & \alpha + \beta + \langle \alpha \rangle \beta & & \gamma \end{array},$$

$$sw_{ux}^{th} : \begin{array}{c|ccc} \omega & h_x & \cdots \\ t_u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|ccc} \omega\epsilon & h_x & \cdots \\ t_u & \alpha(1 + \langle \gamma \rangle / \epsilon) & \beta(1 + \langle \gamma \rangle / \epsilon) \\ \vdots & \gamma / \epsilon & \delta - \gamma\beta / \epsilon \end{array},$$

where $\epsilon := 1 + \alpha$ and $\langle c \rangle := \sum_i c_i$, and let

$$R_{ab}^p := \begin{array}{c|cc} 1 & h_a & h_b \\ t_a & 0 & X - 1 \\ t_b & 0 & 0 \end{array} \quad R_{ab}^m := \begin{array}{c|cc} 1 & h_a & h_b \\ t_a & 0 & X^{-1} - 1 \\ t_b & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.

- Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribboness, cabling, v-knots, knotted graphs, etc., and there’s potential for vast generalizations.

• The least wasteful “Alexander for tangles” I’m aware of.

- Every step along the computation is the invariant of something.
- Fits on one sheet, including implementation & propaganda.



Further meta-monoids. Π (and variants), \mathcal{A} (and quotients), vT , ...

Further meta-bicrossed-products. Π (and variants), $\vec{\mathcal{A}}$ (and quotients), M_0 , M , \mathcal{K}^{bh} , \mathcal{K}^{rbh} , ...

Meta-Lie-algebras. \mathcal{A} (and quotients), \mathcal{S} , ...

Meta-Lie-bialgebras. $\vec{\mathcal{A}}$ (and quotients), ...

I don’t understand the relationship between gr and H , as it appears, for example, in braid theory.

I mean business!

```

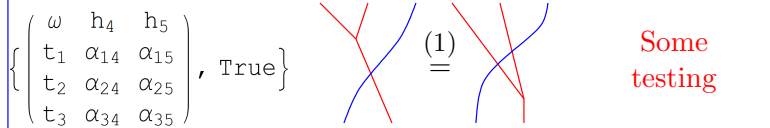
SSimp = Factor; SetAttributes[BCollect, Listable];
BCollect[B[omega_, A_] := B[SSimp[omega],
Collect[A, h, Collect[omega, t, SSimp] &]];
BForm[B[omega_, A_] := Module[{ts, hs, M},
ts = Union[Cases[B[omega, A], t_u -> u, Infinity]];
hs = Union[Cases[B[omega, A], h_x -> x, Infinity]];
M = Outer[SSimp[Coefficient[A, h_x t_u]] &, hs, ts];
PrependTo[M, t_u & /@ ts];
M = Prepend[Transpose[M], Prepend[h_x & /@ hs, omega]];
MatrixForm[M]];
BForm[else_] := else /. B_B -> BForm[B];
Format[B_B, StandardForm] := BForm[B];
    
```

```

<omega_> := # /. t_u -> 1;
tm_u_v_w[omega_] := BCollect[B /. t_u | v -> t_u];
hm_x_y_z[B[omega_, A_] := Module[
{alpha = D[A, h_x], beta = D[A, h_y], gamma = A /. h_x | y -> 0},
B[omega, (alpha + (1 + alpha) beta) h_x + gamma] // BCollect];
sw_u_x[B[omega_, A_] := Module[{alpha, beta, gamma, delta, epsilon},
alpha = Coefficient[A, h_x t_u]; beta = D[A, t_u] /. h_x -> 0;
gamma = D[A, h_x] /. t_u -> 0; delta = A /. h_x | t_u -> 0;
epsilon = 1 + alpha;
B[omega + epsilon, alpha (1 + gamma / epsilon) h_x t_u + beta (1 + gamma / epsilon) t_u
+ gamma / epsilon h_x + delta - gamma * beta / epsilon
] // BCollect];
gm_u_v_w[omega_] := beta // sw_ab // hm_u_v_w // tm_u_v_w;
B /. B[omega_, A_] B[omega_2_, A2_] := B[omega_1 * omega_2, A1 + A2];
Rp_a_b := B[1, (X - 1) t_u h_x];
Rm_a_b := B[1, (X^-1 - 1) t_u h_x];
    
```

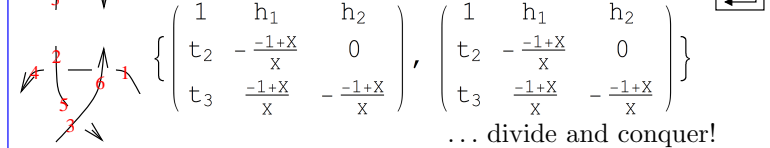
$$\{\beta = \mathbf{B}[\omega, \text{Sum}[\alpha_{10+i+j} t_i h_j, \{\mathbf{i}, \{1, 2, 3\}\}, \{\mathbf{j}, \{4, 5\}\}]\},$$

$$(\beta // tm_{12 \rightarrow 1} // sw_{14}) = (\beta // sw_{24} // sw_{14} // tm_{12 \rightarrow 1})$$



$$\{\mathbf{Rm}_{51} \mathbf{Rm}_{62} \mathbf{Rp}_{34} // gm_{14 \rightarrow 1} // gm_{25 \rightarrow 2} // gm_{36 \rightarrow 3},$$

$$\mathbf{Rp}_{61} \mathbf{Rm}_{24} \mathbf{Rm}_{35} // gm_{14 \rightarrow 1} // gm_{25 \rightarrow 2} // gm_{36 \rightarrow 3}\}$$



$$\beta = \mathbf{Rm}_{12,1} \mathbf{Rm}_{27} \mathbf{Rm}_{83} \mathbf{Rm}_{4,11} \mathbf{Rp}_{16,5} \mathbf{Rp}_{6,13} \mathbf{Rp}_{14,9} \mathbf{Rp}_{10,15}$$

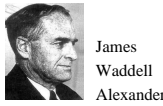
t_1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$-\frac{-1+X}{X}$	0	0	0	0
t_4	0	0	0	0	0	$-\frac{-1+X}{X}$	0	0
t_6	0	0	0	0	0	0	$-1+X$	0
t_8	0	$-\frac{-1+X}{X}$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$-1+X$
t_{12}	$-\frac{-1+X}{X}$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$-1+X$	0	0	0
t_{16}	0	0	$-1+X$	0	0	0	0	0

$$Do[\beta = \beta // gm_{1k \rightarrow 1}, \{k, 2, 10\}]; \beta$$

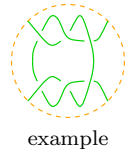
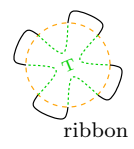
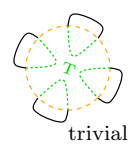
$\frac{1}{X}$	h_1	h_{11}	h_{13}	h_{15}
t_1	$-\frac{(-1+X)(1+X)}{X}$	$-(1+X)(1-X+X^2)$	$(-1+X)(1-X+X^2)$	$-1+X$
t_{12}	$-\frac{-1+X}{X}$	0	0	0
t_{14}	$-1+X$	$\frac{(-1+X)^2(1-X+X^2)}{X}$	$-\frac{(-1+X)^2(1-X+X^2)}{X}$	0
t_{16}	$\frac{-1+X}{X}$	$(-1+X)^2$	$-\frac{(-1+X)^3}{X}$	0

$$Do[\beta = \beta // gm_{1k \rightarrow 1}, \{k, 11, 16\}]; \beta$$

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3} \right)$$



1. Where does it *more simply* come from?
2. Remove all the denominators.
3. How do determinants arise in this context?
4. Understand links (“meta-conjugacy classes”).
5. Find the “reality condition”.
6. Do some “Algebraic Knot Theory”.
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.



"God created the knots, all else in topology is the work of mortals."
 Leopold Kronecker (modified)

www.katlas.org The Knot Atlas
 Injere Car Ete



The Kashiwara-Vergne Problem and Topology

Abstract. I will describe a general machine, a close cousin of Taylor's theorem, whose inputs are topics in topology and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV , solved Alekseev-Meinrenken 2006 ω/AM , elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that $x+y - \log e^y e^x = (1 - e^{-\text{ad } x})F + (e^{\text{ad } y} - 1)G$ in FL and so that with $z = \log e^x e^y$,

$$\text{tr}(\text{ad } x)\partial_x F + \text{tr}(\text{ad } y)\partial_y G \text{ in cyclic words} \\ = \frac{1}{2} \text{tr} \left(\frac{\text{ad } x}{e^{\text{ad } x} - 1} + \frac{\text{ad } y}{e^{\text{ad } y} - 1} - \frac{\text{ad } z}{e^{\text{ad } z} - 1} - 1 \right)$$

Implies the loosely-stated **convolutions statement**: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.

The Machine. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i : a_i \in \mathbb{Q}, g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i : \sum a_i = 0\} \subset \mathcal{K}$ its augmentation ideal. Let

$$\mathcal{A} = \text{gr } \mathcal{K} := \bigoplus_{m \geq 0} \mathcal{I}^m / \mathcal{I}^{m+1}.$$

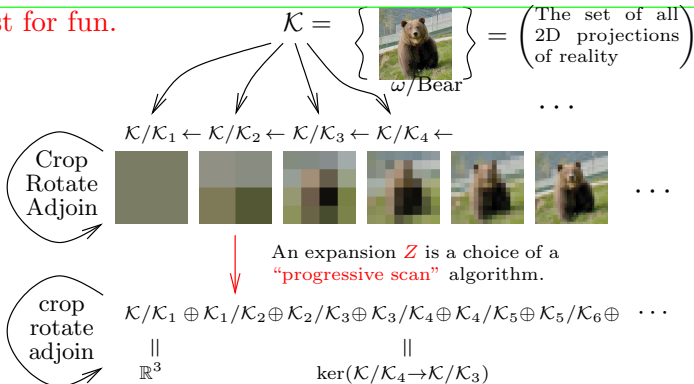
P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that \mathcal{A} inherits a product from G .

Definition. A linear $Z: \mathcal{K} \rightarrow \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0, \dots, 0, \gamma/\mathcal{I}^{m+1}, *, \dots)$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f : f(0) = 0\}$. Then $\mathcal{I}^m = \{f : f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is a homomorphic expansion!

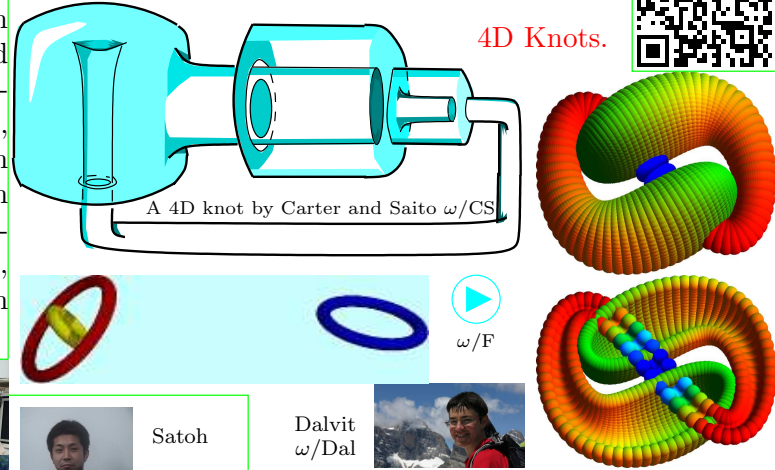
Just for fun.



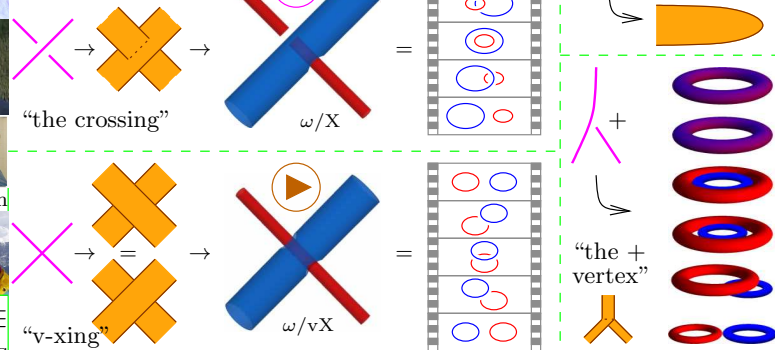
In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. **This is the tip of a major iceberg.**

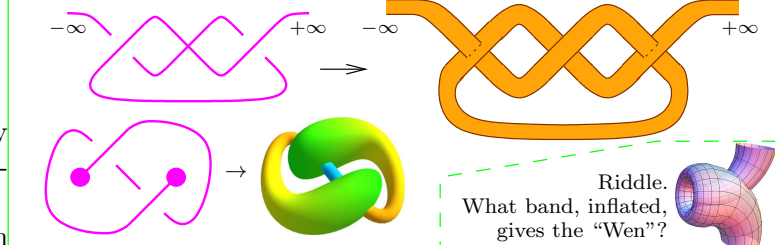
Dancso, ω/ZD



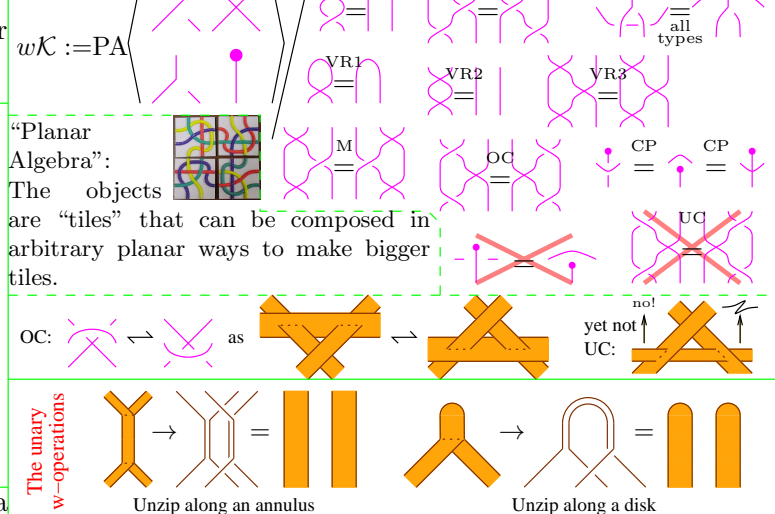
The Generators



The Double Inflation Procedure.

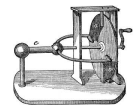


wKO.



The Machine generalizes to arbitrary algebraic structures!

ω/mac

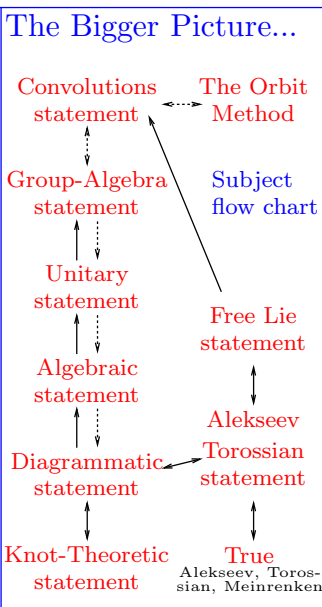


"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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What are w-Trivalent Tangles? (PA := Planar Algebra)

$\{ \text{knots} \} = \text{PA} \langle \text{R123} : \text{ } \rangle$
 $\{ \text{trivalent tangles} \} = \text{PA} \langle \text{R23, R4} : \text{ } \rangle$
 $w\text{TT} = \{ \text{trivalent w-tangles} \} = \text{PA} \langle \text{w-generators} \mid \text{w-relations} \mid \text{unary w-operations} \rangle$

The w-generators.

Broken surface
 Crossing → Broken surface → 2D Symbol → Dim. reduc. → Virtual crossing → Movie



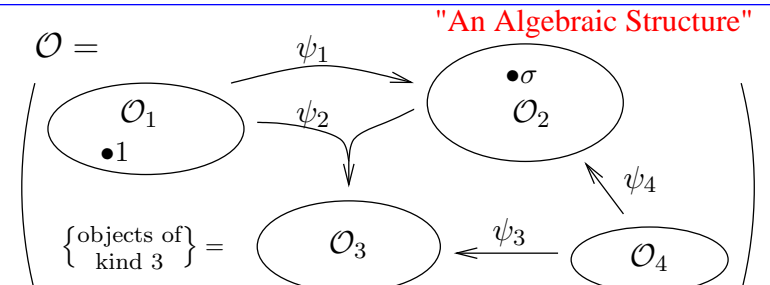
Homomorphic expansions for a filtered algebraic structure \mathcal{K} :

$$\text{ops}^{\leftarrow} \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\text{ops}^{\leftarrow} \text{gr } \mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An **expansion** is a filtration $Z : \mathcal{K} \rightarrow \text{gr } \mathcal{K}$ that "covers" the identity on $\text{gr } \mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any \mathcal{K} , allow formal linear combinations, let \mathcal{K}_1 be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

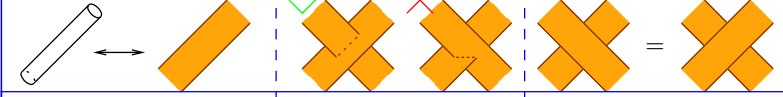
Example: Pure Braids. PB_n is generated by x_{ij} , "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \text{gr } PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n , including the Drinfel'd theory of associators.

Our case(s).

$\mathcal{K} \xrightarrow{Z: \text{high algebra}} \mathcal{A} := \text{gr } \mathcal{K} \xrightarrow{\text{given a "Lie" algebra } \mathfrak{g}} \mathcal{U}(\mathfrak{g})$
 solving finitely many equations in finitely many unknowns low algebra: pictures represent formulas

\mathcal{K} is knot theory or topology; $\text{gr } \mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

A **Ribbon 2-Knot** is a surface S embedded in \mathbb{R}^4 that bounds an immersed handlebody B , with only "ribbon singularities"; a ribbon singularity is a disk D of transverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The **w-relations** include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:

OC: as yet not UC: **Challenge.** Do the Reidemeister!

The unary w-operations

Unzip along an annulus: =
 Unzip along a disk: =

Just for fun.

$\mathcal{K} = \{ \text{Dror Bar-Natan} \} = \left(\text{The set of all b/w 2D projections of reality} \right)$
 $\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \dots$

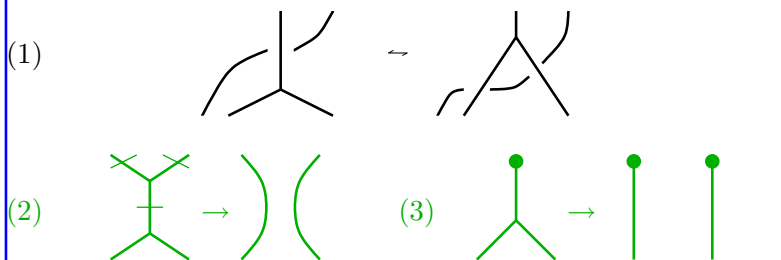
Crop Rotate Adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \dots$
 \parallel
 $\mathbb{R} \parallel \ker(\mathcal{K}/\mathcal{K}_4 \rightarrow \mathcal{K}/\mathcal{K}_3)$

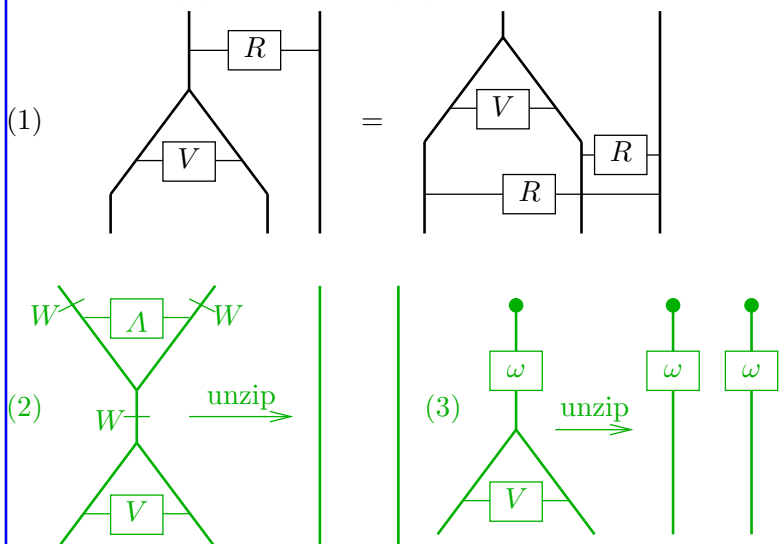
An expansion Z is a choice of a "progressive scan" algorithm.

Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \uparrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{U}(I\mathfrak{g}) \rightarrow \hat{U}(I\mathfrak{g})/\hat{U}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{U}(I\mathfrak{g})$, with W the automorphism of $\hat{U}(I\mathfrak{g})$ induced by flipping the sign of \mathfrak{g}^* , with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{U}(I\mathfrak{g}) \otimes \hat{U}(\mathfrak{g})$ there exist $\omega \in \hat{S}(\mathfrak{g}^*)$ and $V \in \hat{U}(I\mathfrak{g})^{\otimes 2}$ so that

- (1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{U}(I\mathfrak{g})^{\otimes 2} \otimes \hat{U}(\mathfrak{g})$
- (2) $V \cdot SWV = 1$
- (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \text{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

- (1) $V\widehat{e^{x+y}} = \widehat{e^x e^y} V$ (allowing $\hat{U}(\mathfrak{g})$ -valued functions)
- (2) $VV^* = I$
- (3) $V\omega_{x+y} = \omega_x \omega_y$

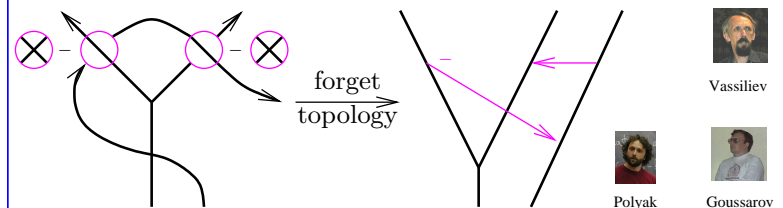
Group-Algebra statement. There exists $\omega^2 \in \text{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{U}(\mathfrak{g})$:

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_{x+y}^2 e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_x^2 \omega_y^2 e^x e^y. \quad (\text{shhh, this is Duflo})$$

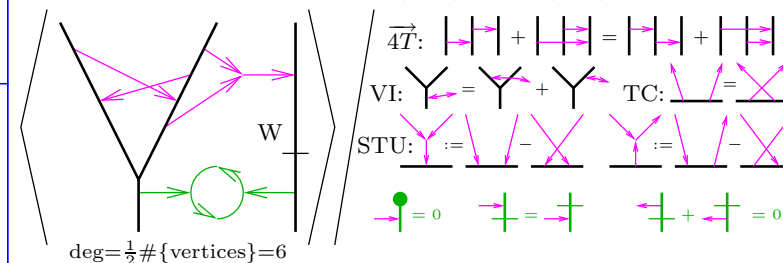
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $j : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

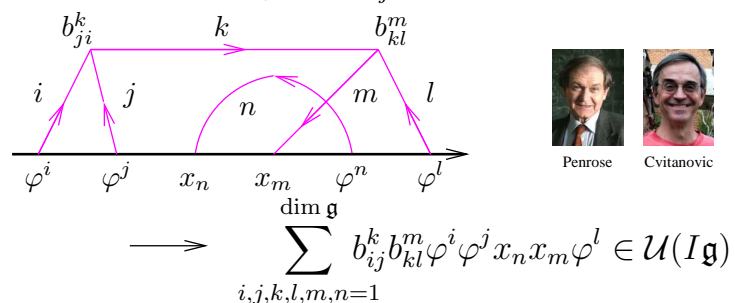
From wTT to \mathcal{A}^w . $\text{gr}_m \text{wTT} := \{m\text{-cubes}\} / \{(m+1)\text{-cubes}\}$:



w-Jacobi diagrams and \mathcal{A} . $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via



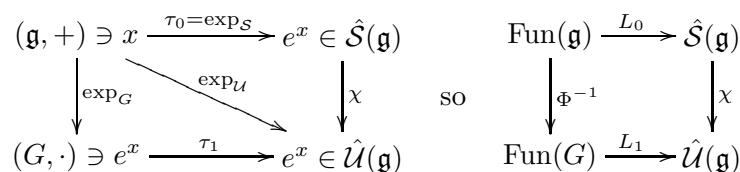
Unitary \iff Algebraic. The key is to interpret $\hat{U}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad } x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{U}(I\mathfrak{g}) \rightarrow \hat{U}(I\mathfrak{g})/\hat{U}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$ is "the constant term".

Unitary \implies Group-Algebra.

$$\iint \omega_{x+y}^2 e^{x+y} \phi(x)\psi(y) = \langle \omega_{x+y}, \omega_{x+y} e^{x+y} \phi(x)\psi(y) \rangle = \langle V\omega_{x+y}, V e^{x+y} \phi(x)\psi(y)\omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y V \phi(x)\psi(y)\omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y \phi(x)\psi(y)\omega_x \omega_y \rangle = \iint \omega_x^2 \omega_y^2 e^x e^y \phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \rightarrow A$ is multiplicative then $(\text{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie (G, \mathfrak{g}) ,



with $L_0\psi = \int \psi(x)e^x dx \in \hat{S}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{U}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{U}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$$\star \text{ in } G : \iint \psi_1(x)\psi_2(y)e^x e^y \quad \star \text{ in } \mathfrak{g} : \iint \psi_1(x)\psi_2(y)e^{x+y}$$

- We skipped...**
- The Alexander polynomial and Milnor numbers.
 - u-Knots, Alekseev-Torossian, and Drinfel'd associators.
 - The simplest problem hyperbolic geometry solves.
 - v-Knots, quantum groups and Etingof-Kazhdan.
 - BF theory and the successful religion of path integrals.