Minsky@Blyth: Complexity in surfaces and 3-manifolds, 1

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Lecture 1. Quasiconformal theory, Thurston's skinning map.

Lecture 2. Synthetic geometry of surfaces.

Lecture 3. Deformation spaces & compactness.

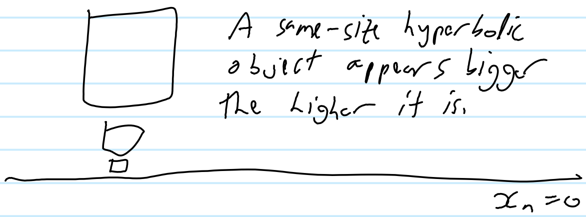
Hyperbolic 3-manifold $:= M = \mathbb{H}^3/\Gamma$

Γ: discrete group of isometries acting freely

$\mathbb{H}^n$: The unique complete simply connected n-manifold with sectional curvature -1,

$= \{(x_1, \dots x_n) : x_n > 0\}$ w/ metric

$$ds^2 = \frac{dx_1^2 + \dots + dx_n^2}{x_n^2}$$



$$S^{n-1}_\infty = [x_n = 0] \cup \{\infty\} \qquad \text{"The sphere at } \infty\text{"}$$

$\mathrm{Iso}\,\mathbb{H}^n$ acts transitively on frames.

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d} \qquad \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{array}$$

"Conformal geometry" at ∞

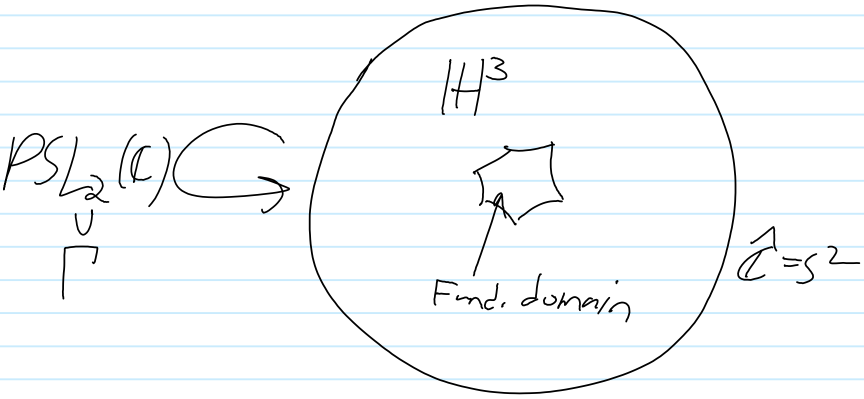
$$\mathrm{Iso}_+\,\mathbb{H}^2 = PSL(2,\mathbb{R})$$

$$\mathbb{H}^3 = \{(z,t) : t > 0\} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}(z,0) = \left(\frac{az+b}{cz+d}, 0\right)$$

$a,b,c,d \in \mathbb{C}$ $\quad ad-bc=1$

$$\mathrm{Isom}_+ \mathbb{H}^3 = PSL(2,\mathbb{C}) \supset PSL(2,\mathbb{R})$$

$\Gamma \subset PSL(2,\mathbb{C})$ e.g. $PSL(2, \mathbb{Z}[i])$ (here $\mathbb{H}^3/\Gamma$ is an "orbifold")



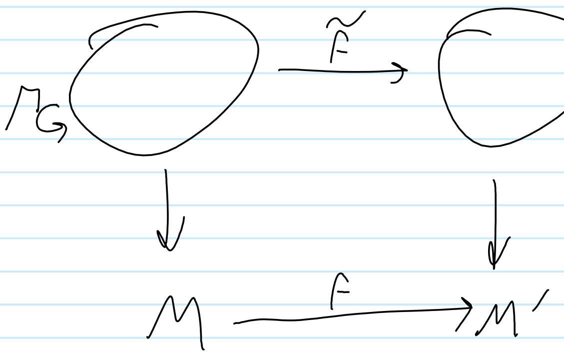
Γ is ergodic on S^2, in fact mixing.

Γ acts on S^2 in "chaotic" way.

Is the hyperbolic structure on M unique?

If M' is diffeomorphic to M, then they are isometric

Sketch of Mostow proof: Assume $f: M \to M'$ is a diffeomorphism. Lift to univ. covers:

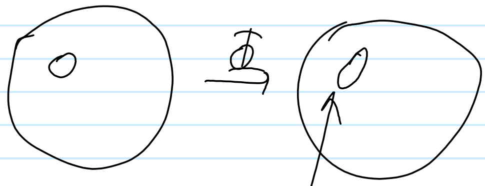


<u>claim</u> $\tilde{f}$ extends to $\Phi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t.

1. $\Gamma' = \Phi \circ \Gamma \circ \Phi^{-1}$
2. Φ is a quasi-conformal homeomorphism

— it has weak derivatives in L^1

— its distortion of the complex structure is bounded: $\left\| \frac{\Phi_{\bar z}}{\Phi_z} \right\|_\infty < k < 1$

$\Longleftrightarrow$ 

bounded eccentricity.

... Get a Γ-invariant, measurable "field of ellipses"

by ergodicity, the ellipses must be circles

$\Rightarrow \Phi$ is conformal

$\Rightarrow \hat{f}$ can be replaced by an isometry.

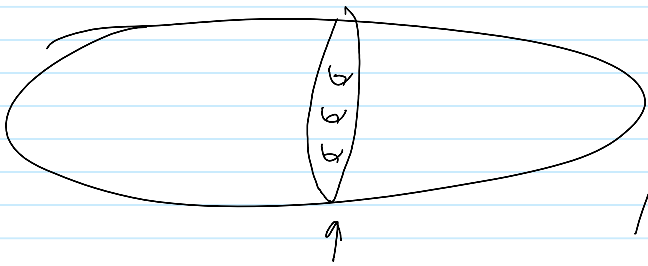
[This works in every dimension > 2]

Prasad extended Mostow to finite-volume 3-manifolds.

Suppose $M = M_1 \cup_\Sigma M_2 \qquad \partial M_1 = \partial M_2 = \Sigma$

$g(\Sigma) > 0$

and $\pi_1(M_i) \hookrightarrow \pi_1(M)$



Σ

$\Gamma = \pi_1(M)$

$\Gamma_i = \pi_1(M_i)$

$\Gamma = \Gamma_1 *_\Sigma \Gamma_2$

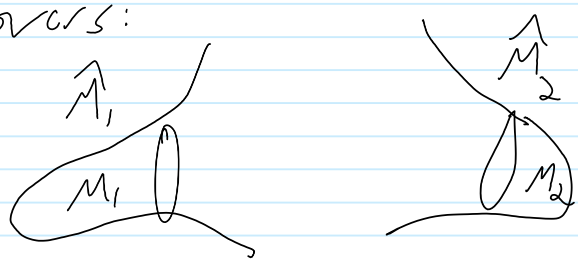
$\pi_1(\Sigma) \hookrightarrow \pi_1(M)$

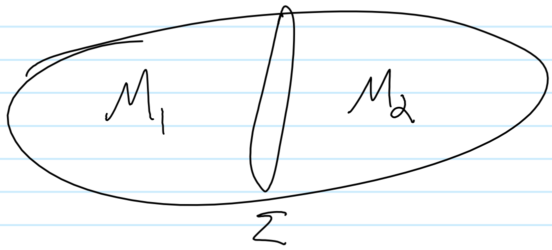
assume also $[\pi_1 M : \pi_1 M_i] = \infty$

have covers:

$\hat{M}_i$

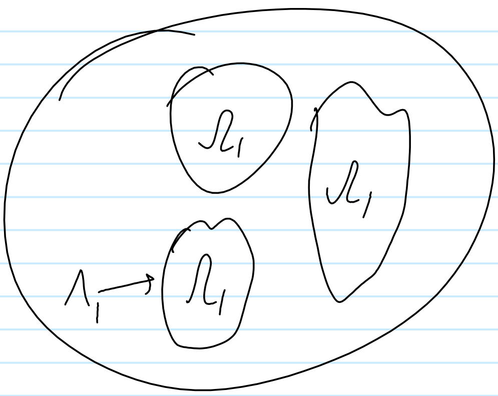
have covers:





Limit set Λ_i = smallest non-empty closed Γ_i-invariant subset of $\hat{\mathbb{C}}$

Regular set $\Omega_i = \hat{\mathbb{C}} \setminus \Lambda_i$



$$\frac{\mathbb{H}^3 \cup \Omega_i}{\Gamma_i} = \hat{M}_i \cup X_i \cong M_i$$

Riem. surface.