Vienna-1402 Handout on February 9, 2014

February-09-14 2:38 PM

symmetry factors?

log

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction to computable combinatorics of perturbative BF theory (Cattaneo-Rossi [CR]), in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. (good news in highlight)

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

With $f\colon (S = \mathbb{R}^2) \to M$, $\xi \in \Omega^0(S, \mathfrak{g})$, $\beta \in \Omega^1(S, \mathfrak{g}^*)$, set

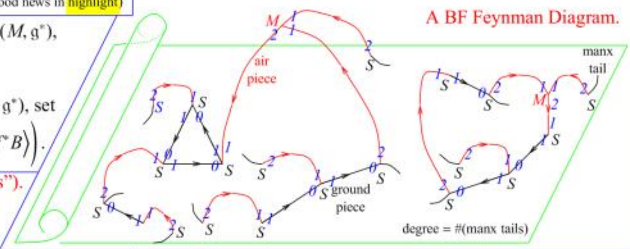
$$O(A,B,f) := \int \mathcal{D}\xi\mathcal{D}\beta \exp\left(\frac{i}{\hbar}\int_S \left\langle \xi, d_{f^*A}\beta + f^*B \right\rangle\right).$$

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(f_i)_{i\in T}$,

$$Z = \sum_{\text{diagrams } D} D \underbrace{\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\text{red } e\in D} \Phi_e^*\omega_3 \prod_{\text{black } e\in D} \Phi_e^*\omega_1$$

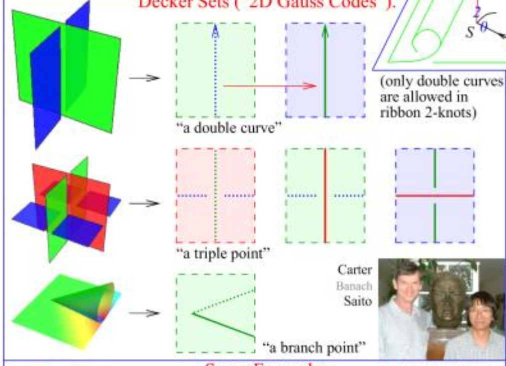
is an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

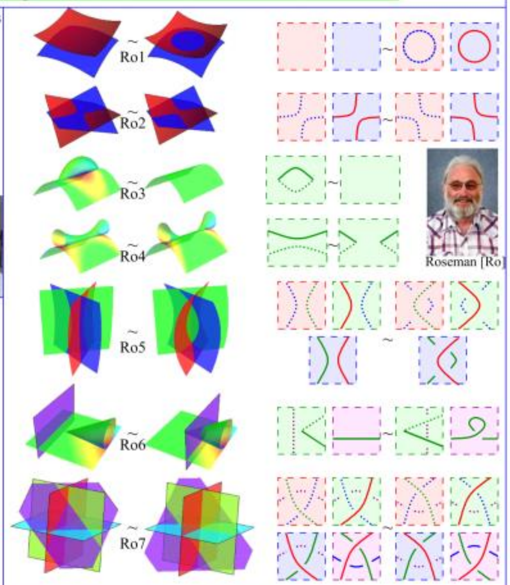
A BF Feynman Diagram.



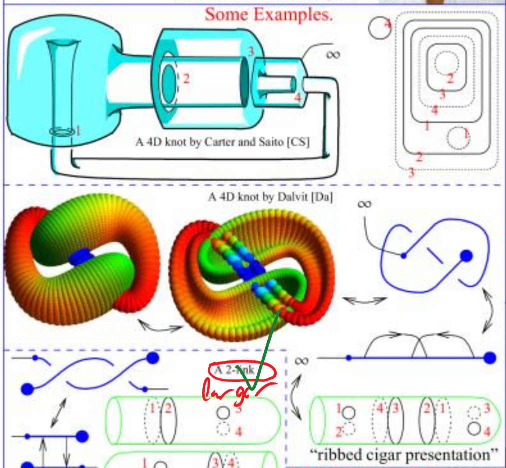
1. consider moving some complexity to the term on the right.
2. Invariance proof?

Decker Sets ("2D Gauss Codes").

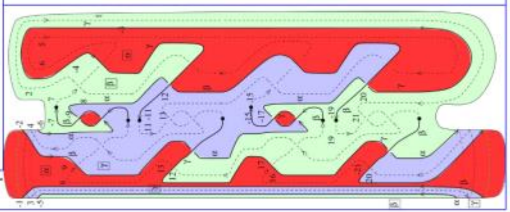




Some Examples.



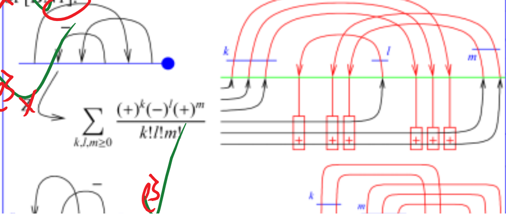
A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].



hyphen

A Partial Reduction of BF Theory to Combinatorics, 2

Theorem. Z of any ribbon 2-knot/link can be computed as follows, and the result agrees with BN-Dancso [BND] and with BN [BN1].

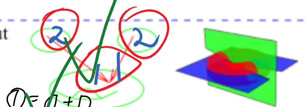


Musings

Chern Simons. When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?

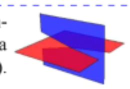
Is this all? What about the ∨-invariant?

(The "true" triple linking number.)

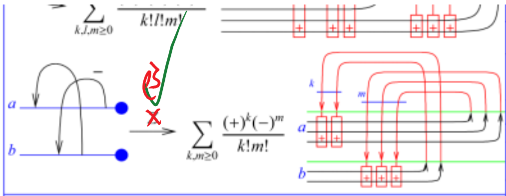


also, enlarge.

Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?).

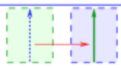


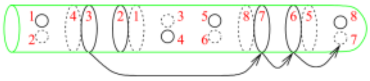
A



B

Sketch of Proof.
In $4D$ axial gauge, only "drop down" red propagators, hence no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious. . .





References.

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[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* `http://www.math.toronto.edu/~drorbn/papers/KBH/`, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots, Invariants of knots and 3-manifolds (Kyoto 2001),* Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* paper, videos (wClips) and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`

[CKS] J. S. Carter, S. Kamada, and M. Saito, *Diagrammatic Computations for Quandles and Cocycle Knot Invariants,* Contemp. Math. **318** (2003) 51–74.

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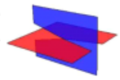
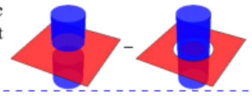
[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology,* Alg. and Geom. Top. **7** (2007) 47-92, arXiv:math/0609742.

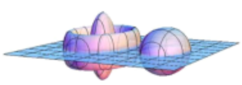
Gnots.
In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?



Finite type.
What are finite-type invariants for 2-knots? What would be "chord diagrams"?

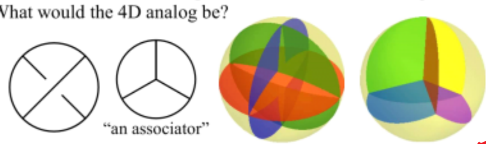


Bubble-wrap-finite-type.
There's an alternative definition of finite type in 3D, due to Goussarov (see [BN2]). The obvious parallel in 4D involves "bubble wraps". Is it any good?



Picture of Goussarov

Shielded tangles.
In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?



"an associator"

a word on KV (make a hardware piece?)

Plane curves.
Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?

	$a(\ast)$	$a(\asymp)$	$a(\asymp)$	∞	$\bigcirc$	$\circledcirc$	◯(oo)	◯(ooo)	$\cdots$
St	1	0	0	0	0	1	2	3	$\cdots$
J^+	0	2	0	0	0	−2	−4	−6	$\cdots$
J^-	0	0	−2	−1	0	−3	−6	−9	$\cdots$

Picture of Arnold.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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Add an explicit computation?

A. consider highlighting the combinatorial construction, perhaps as a "splash opener". Footnotes:

0. Invariant of tangles, vanishes on braids.
1. Contains Alexander.
2. The "missing factor" in Levine's factorization [The rest of Levine's factorization can also easily described in similar terms, hence ∃ contains the MVA.
3. Should be "summed" [interpreted in graph theory] and categorified.