A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction to computable combinatorics of perturbative BF theory (Cattaneo-Rossi [CR]) in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. *(good news in highlight)*

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(f_t)_{t \in T}$,

$$\zeta = \log \sum_{\substack{\text{diagrams} \\ D}} \frac{D}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\text{red } e \in D} \Phi_e^* \omega_3 \prod_{\text{black } e \in D} \Phi_e^* \omega_1$$

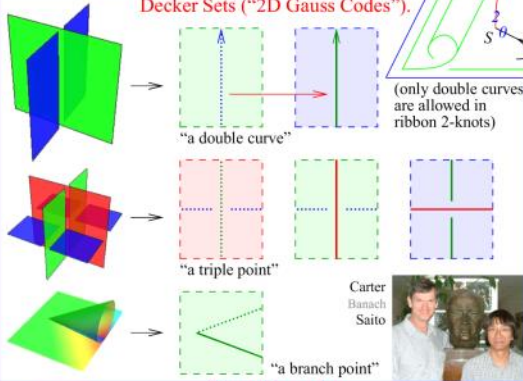is an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,
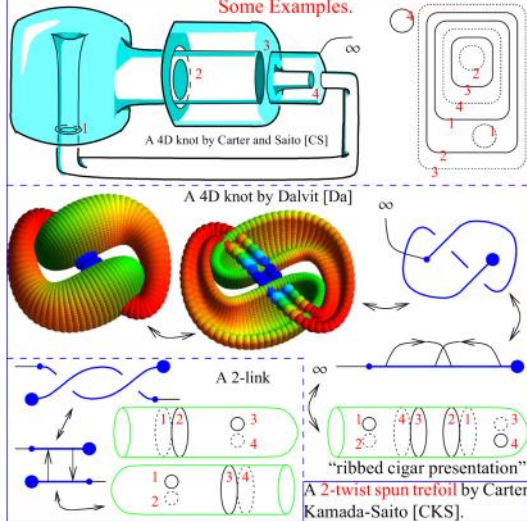
$$S(A, B) := \int_M \langle B, F_A \rangle.$$

With $f : (S = \mathbb{R}^2) \to M$, $\xi \in \Omega^0(S, \mathfrak{g})$, $\beta \in \Omega^1(S, \mathfrak{g}^*)$, set

$$O(A, B, f) := \int \mathcal{D}\xi \mathcal{D}\beta \exp\left(\frac{i}{\hbar}\int_S \langle \xi, d_{f^*A}\beta + f^*B \rangle\right).$$
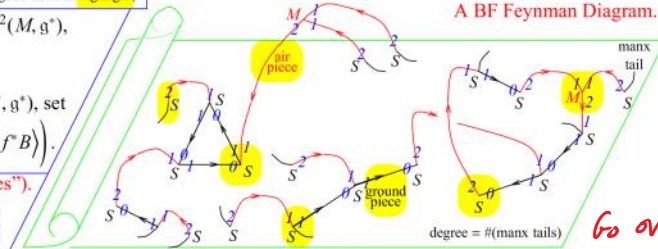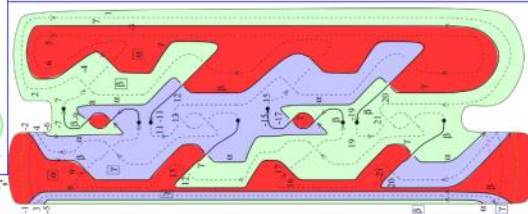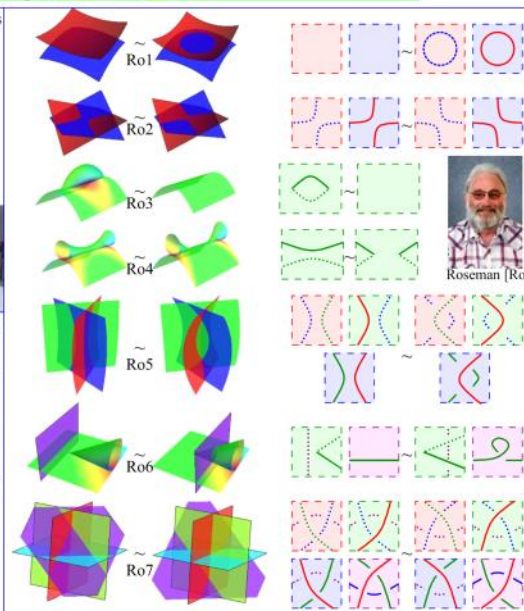
Decker Sets ("2D Gauss Codes").

"a double curve"

(only double curves are allowed in ribbon 2-knots)

"a triple point"

"a branch point"

Carter Banach Saito

A BF Feynman Diagram.

degree = #(manx tails)

(handwritten: Go over invariance proof.)

~Ro1
~Ro2
~Ro3
~Ro4
~Ro5
~Ro6
~Ro7

Roseman [Ro]

Some Examples.

A 4D knot by Carter and Saito [CS]

A 4D knot by Dalvit [Da]

A 2-link

"ribbed cigar presentation"

A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

A Partial Reduction of BF Theory to Combinatorics, 2

Theorem 1 (with Cattaneo (credit, no blame)). In the ribbon case, e^ζ can be computed as follows:

$$e^\zeta \mapsto \sum_{k,l,m \geq 0} \frac{(+)^k(-)^l(+)^m}{k!\,l!\,m!}$$

$$e^\zeta \mapsto \sum_{k,m \geq 0} \frac{(+)^k(-)^m}{k!\,m!}$$

Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

● Agrees with BN-Dancso [BND] and with [BN1]. ● In-practice computable! ● Vanishes on braids. ● Extends to w. ● Contains Alexander. ● The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). ● Related to / extends Farber's [Fa]? ● Should be summed and categorified.

References.

[Ar] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series **5**, American Mathematical Society 1994.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots*, Invariants of knots and 3-manifolds (Kyoto 2001), Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne*, http://www.math.toronto.edu/~drorbn/papers/WKO/.

[CKS] J. S. Carter, S. Kamada, and M. Saito, *Diagrammatic Computations for Quandles and Cocycle Knot Invariants*, Contemp. Math. **318** (2003) 51–74.

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[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links*, Math. Ann. **293** (1992) 543–568.

[Le] J. Levine, *A Factorization of the Conway Polynomial*, Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

[Ro] D. Roseman, *Reidemeister-Type Moves for Surfaces in Four-Dimensional Space*, Knot Theory, Banach Center Publications **42** (1998) 347–380.

[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47-92, arXiv:math/0609742.
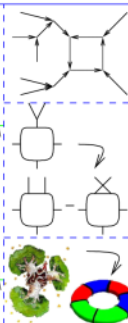
Sketch of Proof. In $4D$ axial gauge, only "drop down" red propagators, hence in the ribbon case, no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious...
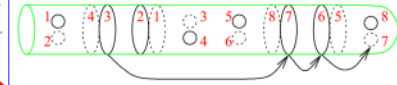
Musings

Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?
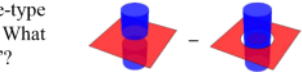
Is this all? What about the ∨-invariant? (the "true" triple linking number)
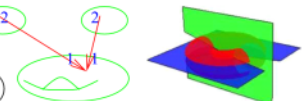
Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?
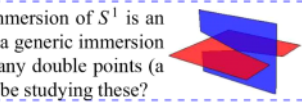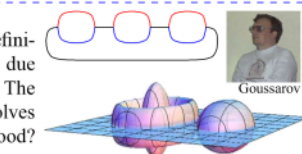
Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?

Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN2]). The obvious parallel in 4D involves "bubble wraps". Is it any good?

Goussarov

Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?

"an associator"

Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?

Arnold

	$a(\text{⌣})$	$a(\text{⤬})$	$a(\text{⤬})$	∞	$\bigcirc$	$\bigcirc\!\!\bigcirc$	$\bigcirc\!\bigcirc$	$\bigcirc\!\bigcirc\!\bigcirc$	$\cdots$
St	1	0	0	0	0	1	2	3	$\cdots$
J^+	0	2	0	0	0	-2	-4	-6	$\cdots$
J^-	0	0	-2	-1	0	-3	-6	-9	$\cdots$

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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virtual 2-knots? Group only?

Find a way to insert YouTubes of Flying vortex Rings & of Knotted Vortices? Maybe in place of "Propaganda"?

https://www.youtube.com/watch?v=mHyTOcfF99o#t=3m45s
http://youtu.be/mHyTOcfF99o#t=3m45s
http://www.youtube.com/watch?v=YCA0VIExVhg&feature=youtu.be#t=50s
http://youtu.be/YCA0VIExVhge#t=50s

Lego picture?

Improve logging at HandoutBrowser.js ? ✓