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February-15-14 1:53 AM

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction to computable combinatorics of perturbative BF theory (Cattaneo-Rossi [CR]), in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting.

(good news in highlight)

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(f_l)_{l\in T}$,

Cattaneo Rossi

credit Cattaneo?

$$\zeta = \log \sum_{\substack{\text{diagrams}\\ D}} \frac{D}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\text{red } e\in D} \Phi_e^*\omega_3 \prod_{\text{black } e\in D} \Phi_e^*\omega_1$$

is an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

Go over invariance proof.

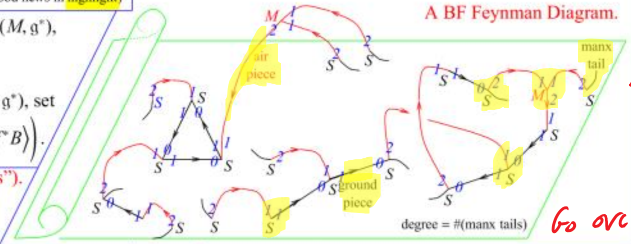
BF Following [CR]. $A \in \Omega^1(M=\mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

With $f\colon (S=\mathbb{R}^2) \to M$, $\xi \in \Omega^0(S,\mathfrak{g})$, $\beta \in \Omega^1(S,\mathfrak{g}^*)$, set

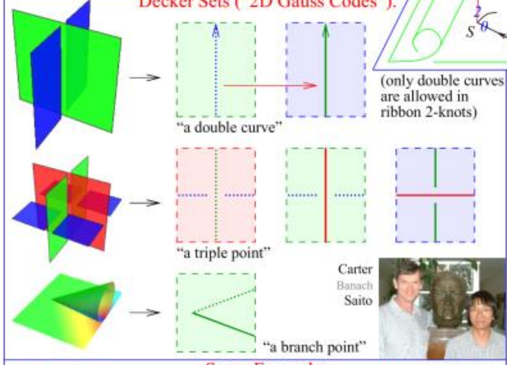
$$O(A,B,f) := \int \mathcal{D}\xi\mathcal{D}\beta \exp\left(\frac{i}{\hbar}\int_S \left\langle \xi, d_{f^*A}\beta + f^*B \right\rangle\right).$$

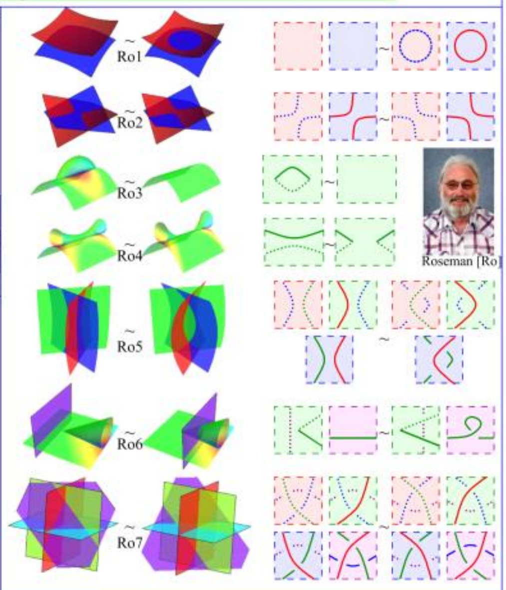
A BF Feynman Diagram.



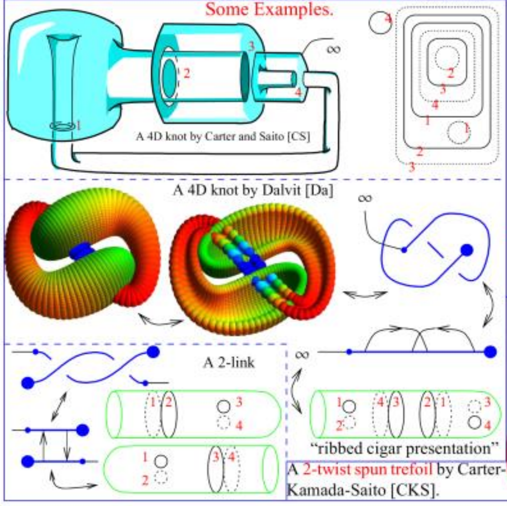
Add highlight?

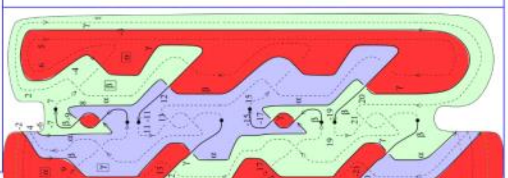
Decker Sets ("2D Gauss Codes").





Some Examples.





A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

replace by an informative picture.

in the ribbon case,

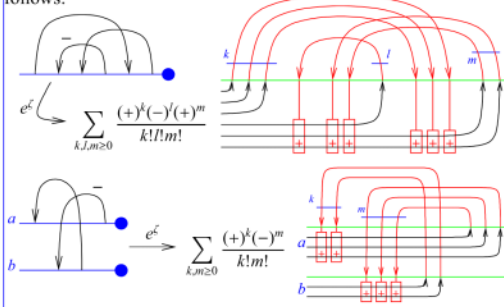
Video and more @ Dror Bar-Natan: Talks: Vienna-1402:
http://www.math.toronto.edu/~drorbn/Talks/Vienna-1402

A Partial Reduction of BF Theory to Combinatorics, 2

Theorem 1. For any ribbon 2-knot/link ... can be computed as ... **Sketch of Proof.** ... In 4D axin the ribbon case,

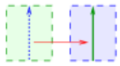
A Partial Reduction of BF Theory to Combinatorics, 2

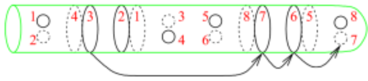
Theorem 1. For any ribbon 2-knot/link, e^{ζ} can be computed as follows:



Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".
• Agrees with BN-Dancso [BND] and with [BN1]. • In-practice computable! • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

Sketch of Proof. In $4D$ axial gauge, only "drop down" red propagators, hence no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious…

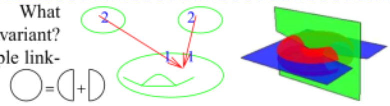




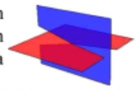
Musings

Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?

Is this all? What about the $\vee$-invariant? (the "true" triple linking number)

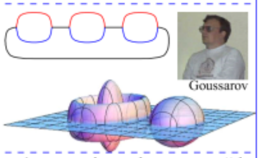


Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?



Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?

Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN2]). The obvious parallel in 4D involves "bubble wraps". Is it any good?



Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?

~~link to~~ ~~Kégs~~ ?

Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

A ~~Lego~~ picture?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?

	$a(\ast)$	$a(\prec)$	$a(\succ)$	∞	○	⊚	∞ inside circle (2 loops)	3 loops	…
St	1	0	0	0	0	1	2	3	…
J^+	0	2	0	0	0	−2	−4	−6	…
J^-	0	0	−2	−1	0	−3	−6	−9	…

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

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virtual 2-knots? Group only?

Can the $\mathbb{R}^4$ gauge group be separated from the $\mathbb{R}^2$ gauge group?