Symmetry factors? (handwritten)

Dror Bar-Natan: Talks: Vienna-1402:
http://www.math.toronto.edu/~drorbn/Talks/Vienna-1402

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction to computable combinatorics of perturbative BF theory (Cattaneo-Rossi [CR]), in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting.

(good news in highlight)

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S^1. Then for a 2-link $(f_t)_{t\in T}$,

$$\zeta = \log \sum_{\text{diagrams } D} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \underbrace{}_{S\text{-vertices}} \int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4} \underbrace{}_{M\text{-vertices}} \prod_{\text{red } e\in D} \Phi_e^*\omega_3 \prod_{\text{black } e\in D} \Phi_e^*\omega_1$$

is an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

Cattaneo   Rossi

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

With $f: (S = \mathbb{R}^2) \to M$, $\xi \in \Omega^0(S, \mathfrak{g})$, $\beta \in \Omega^1(S, \mathfrak{g}^*)$, set

$$O(A, B, f) := \int \mathcal{D}\xi \mathcal{D}\beta \exp\left(\frac{i}{\hbar} \int_S \langle \xi, d_{f^*A}\beta + f^*B \rangle\right).$$

A BF Feynman Diagram.

air piece

ground piece

manx tail

degree = #(manx tails)

1. consider moving some complexity to the term on the right.
2. Invariance proof? (handwritten)

Decker Sets ("2D Gauss Codes").

"a double curve"

(only double curves are allowed in ribbon 2-knots)

"a triple point"

"a branch point"

Carter
Banach
Saito

$\widetilde{Ro}1$
$\widetilde{Ro}2$
$\widetilde{Ro}3$
$\widetilde{Ro}4$
$\widetilde{Ro}5$
$\widetilde{Ro}6$
$\widetilde{Ro}7$

Roseman [Ro]

Some Examples.

A 4D knot by Carter and Saito [CS]

A 4D knot by Dalvit [Da]

A 2-link

"ribbed cigar presentation"

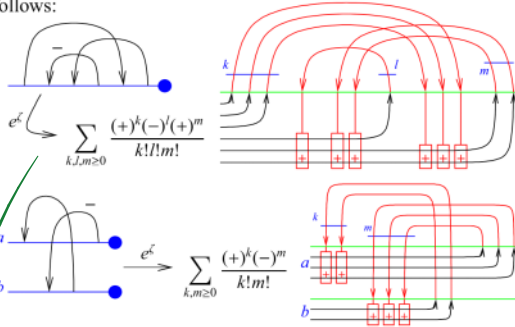A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

A: Add some "collapsing trees on wheels" picture perhaps where marked. (handwritten) ✓

A Partial Reduction of BF Theory to Combinatorics, 2

**Theorem 1.** For any ribbon 2-knot/link, e^ζ can be computed as follows:

$$e^\zeta \curvearrowright \sum_{k,l,m\geq 0} \frac{(+)^k(-)^l(+)^m}{k!\,l!\,m!}$$

$$\xrightarrow{e^\zeta} \sum_{k,m\geq 0} \frac{(+)^k(-)^m}{k!\,m!}$$

Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

• Agrees with BN-Dancso [BND] and with [BN1]. • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

References.

[Ar] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series **5**, American Mathematical Society 1994.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots, Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne*, http://www.math.toronto.edu/~drorbn/papers/WKO/.

[CKS] J. S. Carter, S. Kamada, and M. Saito, *Diagrammatic Computations for Quandles and Cocycle Knot Invariants*, Contemp. Math. **318** (2003) 51–74.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[Da] E. Dalvit, http://science.unitn.it/~dalvit/.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links*, Math. Ann. **293** (1992) 543–568.

[Le] J. Levine, *A Factorization of the Conway Polynomial*, Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

[Ro] D. Roseman, *Reidemeister-Type Moves for Surfaces in Four-Dimensional Space*, Knot Theory, Banach Center Publications **42** (1998) 347–380.

[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47–92, arXiv:math/0609742.

Sketch of Proof. In $4D$ axial gauge, only "drop down" red propagators, hence no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious...


Musings

Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?

Is this all? What about the $\vee$-invariant? (the "true" triple linking number)
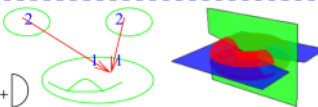

Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?

Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?
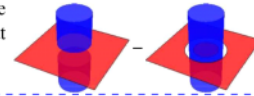
Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN2]). The obvious parallel in 4D involves "bubble wraps". Is it any good?

Goussarov
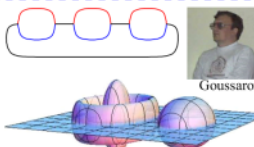
Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?

"an associator"

Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?

Arnold

	$a(\asymp)$	$a(\succ)$	$a(\prec)$	∞	$\bigcirc$	$\bigcirc\hspace$	$\bigcirc\bigcirc$	$\bigcirc\bigcirc\bigcirc$	$\cdots$
St	1	0	0	0	0	1	2	3	$\cdots$
J^+	0	2	0	0	0	-2	-4	-6	$\cdots$
J^-	0	0	-2	-1	0	-3	-6	-9	$\cdots$

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Handwritten annotations:

A√

virtual 2-knots?
Group only?
Add an explicit computation? ∿