

*Symmetry factors?*

**A Partial Reduction of BF Theory to Combinatorics, I**

**Abstract.** I will describe a semi-rigorous reduction to computable combinatorics of perturbative BF theory (Cattaneo-Rossi [CR]), in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. **Weak** this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. *(good news in highlight)*

**The BF Feynman Rules.** For an edge  $e$ , let  $\Phi_e$  be its direction, in  $S^3$  or  $S^1$ . Let  $\omega_3$  and  $\omega_1$  be volume forms on  $S^3$  and  $S^1$ . Then for a 2-link  $(f_i)_{i \in T}$ ,

$$\zeta = \log \sum_{\text{diagrams } D} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4} \prod_{\text{red } e \in D} \Phi_e^* \omega_3 \prod_{\text{black } e \in D} \Phi_e^* \omega_1$$

is an invariant in  $CW(FL(T)) \rightarrow CW(T)$ , "cyclic words in  $T$ ".

**BF Following [CR].**  $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$ ,  $B \in \Omega^2(M, \mathfrak{g}^*)$ ,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

With  $f: (S = \mathbb{R}^2) \rightarrow M$ ,  $\xi \in \Omega^0(S, \mathfrak{g})$ ,  $\beta \in \Omega^1(S, \mathfrak{g}^*)$ , set

$$O(A, B, f) := \int \mathcal{D}\xi \mathcal{D}\beta \exp\left(\frac{i}{\hbar} \int_S \langle \xi, d_f A \beta + f^* B \rangle\right).$$

**Decker Sets ("2D Gauss Codes").**

(only double curves are allowed in ribbon 2-knots)

"a double curve"

"a triple point"

"a branch point"

Carter Banach Saito

**Some Examples.**

A 4D knot by Carter and Saito [CS]

A 4D knot by Dalvit [Da]

A 2-link

"ribbed cigar presentation"

A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

**A BF Feynman Diagram.**

air piece

ground piece

max tail

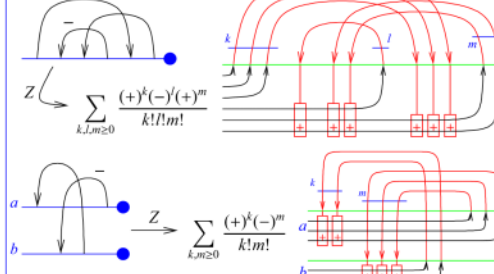
degree = #(max tails)

1. consider moving some complexity to the term on the right.

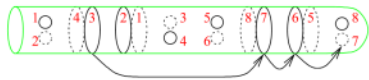
2. Invariance proof?

## A Partial Reduction of BF Theory to Combinatorics, 2

**Theorem.**  $Z = e^{\epsilon}$  of any ribbon 2-knot/link can be computed as follows, and the result agrees with BN-Dancso [BND] and with [BN1].



**Sketch of Proof.** In 4D axial gauge, only "drop down" red propagators, hence no  $M$ -trivalent vertices.  $S$  integrals are  $\pm 1$  iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious...



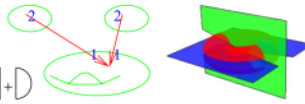
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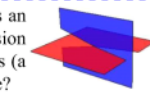
### Musings

**Chern-Simons.** When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?

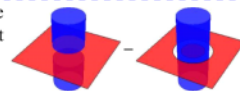
**Is this all?** What about the  $V$ -invariant? (the "true" triple linking number)



**Gnots.** In 3D, a generic immersion of  $S^1$  is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?

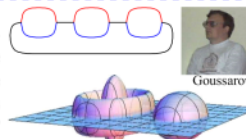


**Finite type.** What are finite-type invariants for 2-knots? What would be "chord diagrams"?



**Bubble-wrap-finite-type.**

There's an alternative definition of finite type in 3D, due to Goussarov (see [BN2]). The obvious parallel in 4D involves "bubble wraps". Is it any good?



**Shielded tangles.** In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?



**Plane curves.** Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's  $J^+$ ,  $J^-$ , and  $St$  [Ar], a bit better?

	$a(\times)$	$a(\infty)$	$a(\infty)$	$\infty$	$\circ$	$\circ$	$\circ$	$\circ$	$\dots$
St	1	0	0	0	0	1	2	3	$\dots$
$J^+$	0	2	0	0	0	-2	-4	-6	$\dots$
$J^-$	0	0	-2	-1	0	-3	-6	-9	$\dots$

virtual 2-knots?  
 Group only?

"God created the knots, all else in topology is the work of mortals."  
 Leopold Kronecker (modified)



a word  
 KV (make a hardware piece?)

Add an explicit computation?

consider highlighting the combinatorial construction, perhaps as a "splash opener". Footnotes:

1. Invariant of tangles, vanishes on braids.
2. Contains Alexander.
3. The "missing factor" in Levine's factorization [The rest of Levine's factorization can also

easily described in similar terms, hence }  
contains the MVA.

3. should be "summed" [interpreted in group  
theory] and categorized.