Rinat Kashaev on Beta pentagon relations

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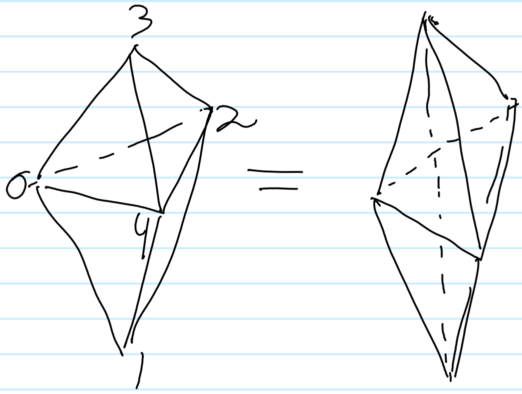
The (quantum) pentagon relation underlies the existing constructions of three dimensional quantum topology in the combinatorial framework of triangulations. Following recent works on constructions of TQFTs with infinite state spaces, I will discuss a special type of integral pentagon relations called beta pentagon relations, and their relationships with the Faddeev type operator pentagon relations.

Motivation: TQFT with ∞-dimensional vector spaces

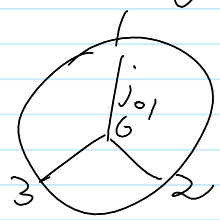
2-3 Pachner Move:

2+3=5 "pentagon"



Biedenharn–Elliott identity on 6j-symbols:

$$\mathbb{C} \ni \begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{Bmatrix} =: T_{0123}$$



$$\sum_{j_{13} \in \mathbb{Z}_{>0}} T_{0123}\, T_{0134}\, T_{1234} = T_{0234}\, T_{0124} \qquad (*)$$

(1) Replace $\mathbb{Z}_{>0}$ by a measure space (X, μ).

(2) Assume the X is a locally compact Abelian group [has Haar measure]

Impose "gauge invariance": $g : \{\text{vertices}\} \to X$

Under $j_{kl} \mapsto j_{kl} + g_k + g_l$, $\{ \vdots \}$ should remain invariance:

$\{\ \ \}$ becomes a map $X^2 \to \mathbb{C}$

(*) Now becomes the β-pentagon identity

$$\begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{21} \end{Bmatrix} \to \varphi(j_{01}+j_{23}-j_{03}-j_{12},\ j_{03}+j_{12}-j_{02}-j_{13})$$

$$\varphi(x,y)\varphi(u,v) = \int \varphi(u+y, v-z)\,\varphi(x+y+u+v-z, z)\,\varphi(x+v, y-z)\,dz$$

Symmetry: $\varphi(x,y) \mapsto \tilde{\varphi}(x,y) = \varphi(-x,-y)$

Suppose $X = \mathbb{R}$, then also a surprising symmetry:

$$\varphi(x,y) \mapsto \tilde{\varphi}(x,y) = \int e^{2\pi i(xv - yu)}\varphi(u,v)\,du\,dv$$

Indices:

$$\varphi_1(x,y)\varphi_3(u,v) = \int \varphi_4(u+y, v-z)\,\varphi_2(x+y+u+v-z, z)\,\varphi_0(x+v, y-z)\,dz$$

Example 1

$$\varphi_j(x,y) = B(a_j + \overset{\sqrt{-1}}{i}(x+y),\ b_j - iy) \qquad a_j, b_j \in \mathbb{R}_{>0}$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$