Alexis Virelizier on 3-dimensional HQFTs

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Idea: TQFT for manifolds M with map $M \to X$, where X is viewed as some "classifying space".

X: fixed pointed topological space.

The Category $HCob_n(X)$:

Objects: $(N, f: N \longrightarrow X)$ N closed pointed (each component) $(n-1)$ mfld

Morphisms: (ori. pres. diff. class of) cobordisms $N_1 \xrightarrow{M} N_2$

$$(N_1, f_1) \xrightarrow{(M,h)} (N_2, f_2)$$

where h is a homotopy class of maps which on the bndry is f_1 / f_2.

$HCob_n(X)$ is symmetric monoidal:

$$\otimes = \sqcup \;, \; 1 = \emptyset$$

An n-dim HQFT w/ target X is a symmetric strong monoidal functor

$$\tau: HCob_n(X) \longrightarrow Vect \text{ or } Mod_k$$

① When $X = pt$, HQFT = TQFT.

② An HQFT produces numerical invariants to any n-dim closed mfld M w/ a class in $[M, X]$.

③ $\tau(-N, f) \cong \tau(N, f)^*$

④ τ induces a f.d. rep. of the mapping class groups of $(n-1)$-mflds anotated by maps $\to X$.

Example Given $\theta \in H^n(X, K^*) \to$ HQFT τ^θ.

$$\tau^\theta(N, f) = K\{a \in C_{n-1}(N) : [a] = [N] \text{ in } H_{n-1}(N)\} / \sim$$

where if $[b] = [a]$ and $b - a = \partial c$ where $c \in C_n(N)$ then $b \sim f^*(\alpha)(c)\, a$ when $\alpha : C_n(X) \to K^*$ represents θ.

1-dim HQFT(X) $\Leftrightarrow$ f.d. dim. rep of $\pi_1(X)$
$\Leftrightarrow$ f.d. flat v. bundle over X.

"HQFT's are higher-dim generalizations of flat vector bundles".

Assume X is aspherical ($\pi_i(X) = 0$ for $i \geq 2$)

$X = K(G, 1)$

$\Rightarrow$ ① if τ is n-dim HQFT(X), $\tau(N, f)$ only depends on the homotopy class of f

② If M^n, $\partial M \neq \emptyset$, $h \in [M, X]$ is a principal

G-bundle over M.

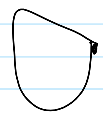
The case $n=2$, $X = K(G,1)$

τ:  $\xrightarrow{\tau} A_{[f]=g}$

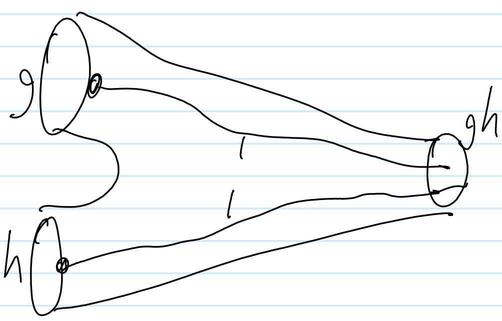
$\downarrow f$

X

denoted

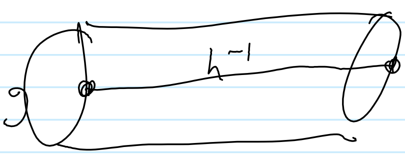
g 

Morphisms:



$\longrightarrow$ map

$A_g \otimes A_h \to A_{gh}$



$\longrightarrow$ map $\varphi: A_g \to A_{hgh^{-1}}$

So $A = \bigoplus A_g$ is a G-graded algebra

w/ action $G \to \mathrm{Aut}(A)$ s.t.

$$a \cdot b = \varphi_h(b)\, a$$

when $a \in A_h$ & $b \in A_k$

The case $n=3$.

Classical: $\mathcal{C}$ spherical fusion category, $\dim \mathcal{C} \neq 0$ $\Rightarrow$ Turaev-Viro TQFT

$\mathcal{B}$ modular Fusion cat $\Rightarrow$ Reshetikhin–Turaev

Thm $\quad RT_{Z(\mathcal{C})} \cong TV_{\mathcal{C}}$

... similar thm in the HQFT case.