Cheat Sheet J - Verification

Pensieve header: Cheat Sheet J Verification; continues pensieve://2013-12/; continued CheatSheetFreeLie-Verification @ pensieve://Projects/WKO4.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2014-01"];
<< FreeLie.m;
IntegrateLieSeries[ls_LieSeries, {s_, s0_, s1_}] :=
  IntegrateLieSeries[ls, {s, s0, s1}] = Module[{ser},
    ser = Unique[IntegrateLieSeries];
    ser[] = Hold[IntegrateLieSeries[ls, {s, s0, s1}]];
    ser[d_Integer] := ser[d] = Expand[Integrate[ls[d], {s, s0, s1}]];
    LieSeries[ser]
   ];
LieSeries /: Integrate[ls_LieSeries, {s_, s0_, s1_}] :=
  IntegrateLieSeries[ls, {s, s0, s1}];
```

```
LieMorphism /: Inverse[mor_LieMorphism] := InvertLieMorphism[mor];
InvertLieMorphism[mor_LieMorphism] := InvertLieMorphism[mor] = Module[{uimg},
    LieMorphism[Table[
      ReleaseHold[Hold[
          uimg[] = Hold[Inverse[mor][u]];
          uimg[1] = u;
          uimg[d_Integer] /; d > 1 := uimg[d] = -Sum[(mor[uimg[k]][d]), {k, 1, d-1}];
          u → LieSeries[uimg]
         ] /. uimg → Unique[InverseLieMorphismOnGenerator]],
      {u, mor[Support]}
     ]]
   ];
```

```
tm[u_, v_, w_] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
CC[us : {___}, γs : {_LieSeries ...}] :=
  LieMorphism[MapThread[Function[{u, γ}, u → Ad[γ][u]], {us, γs}]];
CC[u_, γ_LieSeries] := CC[{u}, {γ}];
CCu_[γ_] := CC[u, γ];
RC[us : {___}, γs : {_LieSeries ...}] := Inverse[CC[us, -γs]];
RC[u_, γ_LieSeries] := RC[{u}, {γ}];
RCu_[γ_] := RC[u, γ];
ad[us : {___}, γs : {_LieSeries ...}] :=
  LieDerivation[MapThread[Function[{u, γ}, u → b[γ, u]], {us, γs}]];
ad[u_, γ_LieSeries] := ad[{u}, {γ}];
adu_[γ_] := ad[u, γ];
(* ad[u_, γ_LieSeries] := LieDerivation[u → b[γ, u]];
adu_[γ_] := ad[u,γ]; *)
ϵ /: ϵ^2 = 0;
Print /@ {{t = ⟨"t"⟩, u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩},
    α = RandomLieSeries[{t, u, v}], δα = RandomLieSeries[{t, u, v}],
    β = RandomLieSeries[{t, u, v}], δβ = RandomLieSeries[{t, u, v}],
    γ = RandomLieSeries[{t, u, v}], δγ = RandomLieSeries[{t, u, v}]
  };
$SeriesShowDegree = 3; $SeriesCompareDegree = 6;
```

{⟨t⟩, ⟨u⟩, ⟨v⟩, ⟨w⟩}

$\mathrm{LS}\Big[-2\,\overline{t}+\overline{u}-2\,\overline{v},\ -\frac{\overline{tu}}{2}+\overline{tv}+\frac{3\,\overline{uv}}{2},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}-2\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}+\overline{\overline{tu}\,u}-\frac{1}{3}\,\overline{\overline{tv}\,u}-\overline{\overline{tv}\,v}+\frac{1}{6}\,\overline{\overline{uv}\,v}\Big]$

$\mathrm{LS}\Big[-\overline{t}-2\,\overline{u}-\overline{v},\ \frac{\overline{tv}}{2}-\frac{3\,\overline{uv}}{2},\ -\overline{t\,\overline{tu}}+2\,\overline{t\,\overline{tv}}-\frac{3}{2}\,\overline{t\,\overline{uv}}-\frac{7}{6}\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{5}{3}\,\overline{\overline{tv}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,v}+\frac{7}{6}\,\overline{\overline{uv}\,v}\Big]$

$\mathrm{LS}\Big[-\overline{t},\ -\overline{tu}+\frac{3\,\overline{tv}}{2}+\frac{\overline{uv}}{2},\ \frac{5}{3}\,\overline{t\,\overline{tu}}-\frac{1}{6}\,\overline{t\,\overline{tv}}-\frac{1}{2}\,\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+\frac{5}{3}\,\overline{\overline{tu}\,u}+\frac{5}{6}\,\overline{\overline{tv}\,u}+\frac{1}{6}\,\overline{\overline{tv}\,v}+\frac{5}{6}\,\overline{\overline{uv}\,v}\Big]$

$\mathrm{LS}\Big[2\,\overline{t}-2\,\overline{u},\ -\overline{tu}+2\,\overline{tv}+2\,\overline{uv},\ -\frac{7}{6}\,\overline{t\,\overline{tu}}+\frac{1}{6}\,\overline{t\,\overline{tv}}-\frac{11}{6}\,\overline{t\,\overline{uv}}-\frac{5}{3}\,\overline{u\,\overline{uv}}-\frac{11}{6}\,\overline{\overline{tu}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,u}-\overline{\overline{tv}\,v}\Big]$

$\mathrm{LS}\Big[-\overline{t}+2\,\overline{u}+\overline{v},\ \overline{tu}+\frac{\overline{tv}}{2}+2\,\overline{uv},\ -\frac{1}{3}\,\overline{t\,\overline{tu}}+\frac{1}{3}\,\overline{t\,\overline{tv}}+\frac{7}{6}\,\overline{t\,\overline{uv}}-\frac{4}{3}\,\overline{u\,\overline{uv}}+\frac{7}{6}\,\overline{\overline{uv}\,v}\Big]$

$\mathrm{LS}\Big[2\,\overline{t}-\overline{v},\ \frac{3\,\overline{tu}}{2}-2\,\overline{tv}+\frac{\overline{uv}}{2},\ \frac{4}{3}\,\overline{t\,\overline{tu}}+\frac{7}{6}\,\overline{t\,\overline{uv}}-\frac{11}{6}\,\overline{u\,\overline{uv}}-\frac{11}{6}\,\overline{\overline{tu}\,u}-\overline{\overline{tv}\,u}-2\,\overline{\overline{tv}\,v}+\frac{2}{3}\,\overline{\overline{uv}\,v}\Big]$

- Some preliminary testing

```
(α // RCu[γ] // CCu[-γ]) ≡ α
```

True

```
(α // RCu[γ] // RCu[-γ // RCu[γ]]) ≡ α
```

True

1. The Definition of J

```
J[u_, γ_] := J[u, γ] = Module[{s}, ∫_0^1 (γ // RC_u[s γ] // div_u // CC_u[-s γ]) ⅆs];
J_u_[γ_] := J[u, γ];
J_u[α][{4}]
```

$$\mathrm{CWS}\Big[-2\,\widehat{u},\ -\widehat{tu}-4\,\widehat{uv},\ -\frac{5\,\widehat{ttu}}{6}-\frac{8\,\widehat{tuu}}{3}-\frac{\widehat{tuv}}{2}-\frac{17\,\widehat{tvu}}{6}-\frac{4\,\widehat{uuv}}{3}-\frac{17\,\widehat{uvv}}{6},$$
$$-\frac{7\,\widehat{tttu}}{4}+\frac{13\,\widehat{ttuu}}{12}+\frac{25\,\widehat{ttuv}}{24}-\frac{\widehat{ttvu}}{8}-\frac{19\,\widehat{tutu}}{6}-\frac{19\,\widehat{tutv}}{4}-\frac{5\,\widehat{tuuu}}{3}+\frac{31\,\widehat{tuuv}}{12}-\frac{25\,\widehat{tuvu}}{3}-$$
$$\frac{3\,\widehat{tuvv}}{4}-\frac{29\,\widehat{tvuu}}{12}-\frac{43\,\widehat{tvuv}}{24}-\frac{43\,\widehat{tvvu}}{24}-2\,\widehat{uuuv}-\frac{81\,\widehat{uuvv}}{8}+\frac{17\,\widehat{uvuv}}{4}-\frac{4\,\widehat{uvvv}}{3}\Big]$$

2. The J_{uv} equation

```
Print /@ {
   0 → {α, β},
   1 → (t1 = J_u[α] + (J_v[β // RC_u[α]] // CC_u[-α])),
   2 → (t2 = J_v[β] + (J_u[α // RC_v[β]] // CC_v[-β])),
   3 → t1 ≡ t2
  };
```

$$0\to\Big\{\mathrm{LS}\Big[\overline{t}-2\,\overline{u}+2\,\overline{v},\ -\overline{tv}+2\,\overline{uv},$$
$$-\frac{1}{2}\,\overline{t\,\overline{tu}}-\frac{11}{6}\,\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{1}{2}\,\overline{\overline{uv}\,v}\Big],$$
$$\mathrm{LS}\Big[2\,\overline{t}-\overline{u}-\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}-\frac{\overline{uv}}{2},\ \frac{1}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}+\frac{1}{2}\,\overline{t\,\overline{uv}}+\frac{2}{3}\,\overline{u\,\overline{uv}}+\frac{3}{2}\,\overline{\overline{tu}\,u}+2\,\overline{\overline{tv}\,u}+\frac{11}{6}\,\overline{\overline{tv}\,v}\Big]\Big\}$$

$$1\to\mathrm{CWS}\Big[-2\,\widehat{u}-\widehat{v},\ -\widehat{tu}-\frac{\widehat{tv}}{2}-2\,\widehat{uv},$$
$$-\frac{5\,\widehat{ttu}}{6}-\frac{5\,\widehat{ttv}}{3}-\frac{8\,\widehat{tuu}}{3}+\frac{13\,\widehat{tuv}}{3}-\frac{21\,\widehat{tvu}}{4}-\frac{23\,\widehat{tvv}}{12}+\frac{29\,\widehat{uuv}}{12}-\frac{11\,\widehat{uvv}}{12}\Big]$$

$$2\to\mathrm{CWS}\Big[-2\,\widehat{u}-\widehat{v},\ -\widehat{tu}-\frac{\widehat{tv}}{2}-2\,\widehat{uv},$$
$$-\frac{5\,\widehat{ttu}}{6}-\frac{5\,\widehat{ttv}}{3}-\frac{8\,\widehat{tuu}}{3}+\frac{13\,\widehat{tuv}}{3}-\frac{21\,\widehat{tvu}}{4}-\frac{23\,\widehat{tvv}}{12}+\frac{29\,\widehat{uuv}}{12}-\frac{11\,\widehat{uvv}}{12}\Big]$$

3 → True

3. The t equation

```
Print /@ {
   0 → γ,
   1 → (t1 = J[w, γ // tm[u, v, w]]),
   2 → (t2 = J_u[γ] // tm[u, v, w]),
   3 → (t3 = J_v[γ // RC_u[γ]] // CC_u[-γ] // tm[u, v, w]),
   4 → t1 ≡ t2 + t3
  };
```

$0 \to \mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{u},\ -2\,\overline{tu} - 2\,\overline{tv},$
$\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{1}{2}\,\overline{t\,\overline{tv}} + \frac{2}{3}\,\overline{t\,\overline{uv}} + \frac{11}{6}\,\overline{u\,\overline{uv}} - \frac{1}{2}\,\overline{\overline{tu}\,u} + \frac{7}{6}\,\overline{\overline{tv}\,u} - \frac{1}{6}\,\overline{\overline{tv}\,v} + \overline{\overline{uv}\,v}\Big]$

$1 \to \mathrm{CWS}\Big[-2\,\widehat{w},\ -2\,\widehat{tw},\ \frac{17\,\widehat{ttw}}{6} - \frac{19\,\widehat{tww}}{6}\Big]$

$2 \to \mathrm{CWS}\Big[-2\,\widehat{w},\ 0,\ \frac{4\,\widehat{ttw}}{3} - \frac{5\,\widehat{tww}}{6} - \frac{5\,\widehat{www}}{6}\Big]$

$3 \to \mathrm{CWS}\Big[0,\ -2\,\widehat{tw},\ \frac{3\,\widehat{ttw}}{2} - \frac{7\,\widehat{tww}}{3} + \frac{5\,\widehat{www}}{6}\Big]$

4 → True

```
Print /@ {
   0 → {γ, γw = γ // tm[u, v, w]},
   1 → (t1 = J[w, γw] // RC_w[γw]),
   2 → (t2 = J_u[γ] // tm[u, v, w] // RC_w[γw]),
   3 → (t3 = J_v[γ // RC_u[γ]] // RC_v[γ // RC_u[γ]] // tm[u, v, w]),
   4 → t1 ≡ t2 + t3
  };
```

$0 \to \Big\{\mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{u},\ -2\,\overline{tu} - 2\,\overline{tv},$
$\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{1}{2}\,\overline{t\,\overline{tv}} + \frac{2}{3}\,\overline{t\,\overline{uv}} + \frac{11}{6}\,\overline{u\,\overline{uv}} - \frac{1}{2}\,\overline{\overline{tu}\,u} + \frac{7}{6}\,\overline{\overline{tv}\,u} - \frac{1}{6}\,\overline{\overline{tv}\,v} + \overline{\overline{uv}\,v}\Big],$
$\mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{w},\ -4\,\overline{tw},\ \frac{1}{6}\,\overline{t\,\overline{tw}} + \frac{1}{2}\,\overline{\overline{tw}\,w}\Big]\Big\}$

$1 \to \mathrm{CWS}\Big[-2\,\widehat{w},\ -2\,\widehat{tw},\ \frac{17\,\widehat{ttw}}{6} - \frac{19\,\widehat{tww}}{6}\Big]$

$2 \to \mathrm{CWS}\Big[-2\,\widehat{w},\ 0,\ \frac{4\,\widehat{ttw}}{3} - \frac{5\,\widehat{tww}}{6} - \frac{5\,\widehat{www}}{6}\Big]$

$3 \to \mathrm{CWS}\Big[0,\ -2\,\widehat{tw},\ \frac{3\,\widehat{ttw}}{2} - \frac{7\,\widehat{tww}}{3} + \frac{5\,\widehat{www}}{6}\Big]$

4 → True

4. The h equation

```
Print /@ {
   1 → (t1 = J[u, BCH[α, β]]),
   2 → (t2 = J[u, α]),
   3 → (t3 = J[u, β // RC[u, α]] // CC[u, -α]),
   4 → t1 ≡ t2 + t3
  };
```

$$1 \to \mathrm{CWS}\left[-3\,\widehat{u},\ -4\,\widehat{tu}-5\,\widehat{uv},\ -\frac{14\,\widehat{ttu}}{3}-\frac{13\,\widehat{tuu}}{2}-\frac{89\,\widehat{tuv}}{12}+\frac{\widehat{tvu}}{4}-\frac{79\,\widehat{uuv}}{12}-\frac{13\,\widehat{uvv}}{4}\right]$$

$$2 \to \mathrm{CWS}\left[-2\,\widehat{u},\ -\widehat{tu}-4\,\widehat{uv},\ -\frac{5\,\widehat{ttu}}{6}-\frac{8\,\widehat{tuu}}{3}-\frac{\widehat{tuv}}{2}-\frac{17\,\widehat{tvu}}{6}-\frac{4\,\widehat{uuv}}{3}-\frac{17\,\widehat{uvv}}{6}\right]$$

$$3 \to \mathrm{CWS}\left[-\widehat{u},\ -3\,\widehat{tu}-\widehat{uv},\ -\frac{23\,\widehat{ttu}}{6}-\frac{23\,\widehat{tuu}}{6}-\frac{83\,\widehat{tuv}}{12}+\frac{37\,\widehat{tvu}}{12}-\frac{21\,\widehat{uuv}}{4}-\frac{5\,\widehat{uvv}}{12}\right]$$

4 → True

- h and S

```
(Plus[
    Ju[γ] // RCu[γ],
    Ju[-γ // RCu[γ]]
   ] // RCu[-γ // RCu[γ]]) @ {6}
```

CWS[0, 0, 0, 0, 0, 0]

5. The meaning(s) of RC

```
Print /@ {
   1 → α,
   2 → (t1 = α // CC[u, γ] // RC[u, -γ]),
   3 → α ≡ t1
  };
```

$$1 \to \mathrm{LS}\Big[\overline{t}-2\,\overline{u}+2\,\overline{v},\ -\overline{tv}+2\,\overline{uv},$$
$$-\frac{1}{2}\,\overline{t\,\overline{tu}}-\frac{11}{6}\,\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{1}{2}\,\overline{\overline{uv}\,v}\Big]$$

$$2 \to \mathrm{LS}\Big[\overline{t}-2\,\overline{u}+2\,\overline{v},\ -\overline{tv}+2\,\overline{uv},$$
$$-\frac{1}{2}\,\overline{t\,\overline{tu}}-\frac{11}{6}\,\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{1}{2}\,\overline{\overline{uv}\,v}\Big]$$

3 → True

```
Print /@ {
   1 → α,
   2 → (t1 = α // CC[u, γ // RC[u, γ]]),
   3 → (t2 = α // RC[u, γ]),
   4 → t1 ≡ t2
  };
```

$$1 \to \mathrm{LS}\Big[\overline{t} - 2\,\overline{u} + 2\,\overline{v},\ -\overline{tv} + 2\,\overline{uv},$$
$$-\frac{1}{2}\,\overline{t\,\overline{tu}} - \frac{11}{6}\,\overline{t\,\overline{tv}} + \overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - \frac{7}{6}\,\overline{\overline{tv}\,u} + \frac{1}{3}\,\overline{\overline{tv}\,v} + \frac{1}{2}\,\overline{\overline{uv}\,v}\Big]$$

$$2 \to \mathrm{LS}\Big[\overline{t} - 2\,\overline{u} + 2\,\overline{v},\ 4\,\overline{tu} - \overline{tv} + 2\,\overline{uv},$$
$$-\frac{9}{2}\,\overline{t\,\overline{tu}} - \frac{11}{6}\,\overline{t\,\overline{tv}} - 3\,\overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - \frac{7}{6}\,\overline{\overline{tv}\,u} + \frac{1}{3}\,\overline{\overline{tv}\,v} + \frac{1}{2}\,\overline{\overline{uv}\,v}\Big]$$

$$3 \to \mathrm{LS}\Big[\overline{t} - 2\,\overline{u} + 2\,\overline{v},\ 4\,\overline{tu} - \overline{tv} + 2\,\overline{uv},$$
$$-\frac{9}{2}\,\overline{t\,\overline{tu}} - \frac{11}{6}\,\overline{t\,\overline{tv}} - 3\,\overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - \frac{7}{6}\,\overline{\overline{tv}\,u} + \frac{1}{3}\,\overline{\overline{tv}\,v} + \frac{1}{2}\,\overline{\overline{uv}\,v}\Big]$$

4 → True

6. $C_u\, C_v$ and $RC_u\, RC_v$

```
Print /@ {
   1 → {α, β, γ},
   2 → (t1 = γ // CC_u[α // RC_v[-β]] // CC_v[β]),
   3 → (t2 = γ // CC_v[β // RC_u[-α]] // CC_u[α]),
   4 → t1 ≡ t2
  };
```

$$1 \to \Big\{\mathrm{LS}\Big[\overline{t} - 2\,\overline{u} + 2\,\overline{v},\ -\overline{tv} + 2\,\overline{uv},$$
$$-\frac{1}{2}\,\overline{t\,\overline{tu}} - \frac{11}{6}\,\overline{t\,\overline{tv}} + \overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - \frac{7}{6}\,\overline{\overline{tv}\,u} + \frac{1}{3}\,\overline{\overline{tv}\,v} + \frac{1}{2}\,\overline{\overline{uv}\,v}\Big],$$
$$\mathrm{LS}\Big[2\,\overline{t} - \overline{u} - \overline{v},\ -\overline{tu} + \frac{\overline{tv}}{2} - \frac{\overline{uv}}{2},\ \frac{1}{3}\,\overline{t\,\overline{tu}} - \frac{3}{2}\,\overline{t\,\overline{tv}} + \frac{1}{2}\,\overline{t\,\overline{uv}} + \frac{2}{3}\,\overline{u\,\overline{uv}} + \frac{3}{2}\,\overline{\overline{tu}\,u} + 2\,\overline{\overline{tv}\,u} + \frac{11}{6}\,\overline{\overline{tv}\,v}\Big],$$
$$\mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{u},\ -2\,\overline{tu} - 2\,\overline{tv},$$
$$\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{1}{2}\,\overline{t\,\overline{tv}} + \frac{2}{3}\,\overline{t\,\overline{uv}} + \frac{11}{6}\,\overline{u\,\overline{uv}} - \frac{1}{2}\,\overline{\overline{tu}\,u} + \frac{7}{6}\,\overline{\overline{tv}\,u} - \frac{1}{6}\,\overline{\overline{tv}\,v} + \overline{\overline{uv}\,v}\Big]\Big\}$$

$$2 \to \mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{u},\ -4\,\overline{tu} - 2\,\overline{tv} + 4\,\overline{uv},$$
$$-\frac{7}{3}\,\overline{t\,\overline{tu}} - \frac{9}{2}\,\overline{t\,\overline{tv}} + \frac{32}{3}\,\overline{t\,\overline{uv}} + \frac{11}{6}\,\overline{u\,\overline{uv}} - \frac{5}{2}\,\overline{\overline{tu}\,u} + \frac{31}{6}\,\overline{\overline{tv}\,u} - \frac{1}{6}\,\overline{\overline{tv}\,v} - 3\,\overline{\overline{uv}\,v}\Big]$$

$$3 \to \mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{u},\ -4\,\overline{tu} - 2\,\overline{tv} + 4\,\overline{uv},$$
$$-\frac{7}{3}\,\overline{t\,\overline{tu}} - \frac{9}{2}\,\overline{t\,\overline{tv}} + \frac{32}{3}\,\overline{t\,\overline{uv}} + \frac{11}{6}\,\overline{u\,\overline{uv}} - \frac{5}{2}\,\overline{\overline{tu}\,u} + \frac{31}{6}\,\overline{\overline{tv}\,u} - \frac{1}{6}\,\overline{\overline{tv}\,v} - 3\,\overline{\overline{uv}\,v}\Big]$$

4 → True

```
Print /@ {
   1 → {α, β, γ},
   2 → (t1 = γ // RC_u[α] // RC_v[β // RC_u[α]]),
   3 → (t2 = γ // RC_v[β] // RC_u[α // RC_v[β]]),
   4 → t1 ≡ t2
  };
```

1 → {LS[$\overline{t} - 2\,\overline{u} + 2\,\overline{v}$, $-\overline{tv} + 2\,\overline{uv}$,
$-\frac{1}{2}\,\overline{t\overline{tu}} - \frac{11}{6}\,\overline{t\overline{tv}} + \overline{t\overline{uv}} - 2\,\overline{u\overline{uv}} + 2\,\overline{\overline{tu}u} - \frac{7}{6}\,\overline{\overline{tv}u} + \frac{1}{3}\,\overline{\overline{tv}v} + \frac{1}{2}\,\overline{\overline{uv}v}$],
LS[$2\,\overline{t} - \overline{u} - \overline{v}$, $-\overline{tu} + \frac{\overline{tv}}{2} - \frac{\overline{uv}}{2}$, $\frac{1}{3}\,\overline{t\overline{tu}} - \frac{3}{2}\,\overline{t\overline{tv}} + \frac{1}{2}\,\overline{t\overline{uv}} + \frac{2}{3}\,\overline{u\overline{uv}} + \frac{3}{2}\,\overline{\overline{tu}u} + 2\,\overline{\overline{tv}u} + \frac{11}{6}\,\overline{\overline{tv}v}$],
LS[$-2\,\overline{t} - 2\,\overline{u}$, $-2\,\overline{tu} - 2\,\overline{tv}$,
$\frac{2}{3}\,\overline{t\overline{tu}} - \frac{1}{2}\,\overline{t\overline{tv}} + \frac{2}{3}\,\overline{t\overline{uv}} + \frac{11}{6}\,\overline{u\overline{uv}} - \frac{1}{2}\,\overline{\overline{tu}u} + \frac{7}{6}\,\overline{\overline{tv}u} - \frac{1}{6}\,\overline{\overline{tv}v} + \overline{\overline{uv}v}$]}

2 → LS[$-2\,\overline{t} - 2\,\overline{u}$, $-4\,\overline{tu} - 2\,\overline{tv} + 4\,\overline{uv}$,
$-\frac{7}{3}\,\overline{t\overline{tu}} - \frac{9}{2}\,\overline{t\overline{tv}} + \frac{32}{3}\,\overline{t\overline{uv}} + \frac{35}{6}\,\overline{u\overline{uv}} + \frac{3}{2}\,\overline{\overline{tu}u} - \frac{17}{6}\,\overline{\overline{tv}u} - \frac{1}{6}\,\overline{\overline{tv}v} - 3\,\overline{\overline{uv}v}$]

3 → LS[$-2\,\overline{t} - 2\,\overline{u}$, $-4\,\overline{tu} - 2\,\overline{tv} + 4\,\overline{uv}$,
$-\frac{7}{3}\,\overline{t\overline{tu}} - \frac{9}{2}\,\overline{t\overline{tv}} + \frac{32}{3}\,\overline{t\overline{uv}} + \frac{35}{6}\,\overline{u\overline{uv}} + \frac{3}{2}\,\overline{\overline{tu}u} - \frac{17}{6}\,\overline{\overline{tv}u} - \frac{1}{6}\,\overline{\overline{tv}v} - 3\,\overline{\overline{uv}v}$]

4 → True

7.

8.

9.

10.

11. div property uv

```
Print /@ {
    0 → {α, β},
    1 → (t1 = div_u[α] // ad_v[β]),
    2 → (t2 = div_v[β] // ad_u[α]),
    3 → (t3 = MakeCWSeries[0]),
    4 → (t4 = div_u[α // ad_v[β]]),
    5 → (t5 = div_v[β // ad_u[α]]),
    6 → t1 - t2 ≡ t3 + t4 - t5
   };
```

$0\to\Big\{\mathrm{LS}\Big[\overline{t}-2\,\overline{u}+2\,\overline{v},\ -\overline{tv}+2\,\overline{uv},$
$-\frac{1}{2}\,\overline{t\,\overline{tu}}-\frac{11}{6}\,\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{1}{2}\,\overline{\overline{uv}\,v}\Big],$
$\mathrm{LS}\Big[2\,\overline{t}-\overline{u}-\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}-\frac{\overline{uv}}{2},\ \frac{1}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}+\frac{1}{2}\,\overline{t\,\overline{uv}}+\frac{2}{3}\,\overline{u\,\overline{uv}}+\frac{3}{2}\,\overline{\overline{tu}\,u}+2\,\overline{\overline{tv}\,u}+\frac{11}{6}\,\overline{\overline{tv}\,v}\Big]\Big\}$

$1\to\mathrm{CWS}\big[0,\ 0,\ 4\,\widehat{tuv}-4\,\widehat{tvu}\big]$

$2\to\mathrm{CWS}\Big[0,\ 0,\ -\frac{\widehat{tuv}}{2}+\frac{\widehat{tvu}}{2}\Big]$

$3\to\mathrm{CWS}[0,\ 0,\ 0]$

$4\to\mathrm{CWS}\big[0,\ 2\,\widehat{uv},\ 6\,\widehat{tuv}-5\,\widehat{tvu}+2\,\widehat{uuv}-\widehat{uvv}\big]$

$5\to\mathrm{CWS}\Big[0,\ 2\,\widehat{uv},\ \frac{3\,\widehat{tuv}}{2}-\frac{\widehat{tvu}}{2}+2\,\widehat{uuv}-\widehat{uvv}\Big]$

6 → True

12. div property uu

```
Print /@ {
   0 → {α, β},
   1 → (t1 = div_u[α] // ad_u[β]),
   2 → (t2 = div_u[β] // ad_u[α]),
   3 → (t3 = div_u[b[α, β]]),
   4 → (t4 = div_u[α // ad_u[β]]),
   5 → (t5 = div_u[β // ad_u[α]]),
   6 → t1 - t2 ≡ t3 + t4 - t5
  };
```

$0\to\Big\{\mathrm{LS}\Big[\overline{t}-2\,\overline{u}+2\,\overline{v},\ -\overline{tv}+2\,\overline{uv},$
$-\frac{1}{2}\,\overline{t\,\overline{tu}}-\frac{11}{6}\,\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{1}{2}\,\overline{\overline{uv}\,v}\Big],$
$\mathrm{LS}\Big[2\,\overline{t}-\overline{u}-\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}-\frac{\overline{uv}}{2},\ \frac{1}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}+\frac{1}{2}\,\overline{t\,\overline{uv}}+\frac{2}{3}\,\overline{u\,\overline{uv}}+\frac{3}{2}\,\overline{\overline{tu}\,u}+2\,\overline{\overline{tv}\,u}+\frac{11}{6}\,\overline{\overline{tv}\,v}\Big]\Big\}$

$1\to\mathrm{CWS}\big[0,\ 0,\ -4\,\widehat{tuv}+4\,\widehat{tvu}\big]$

$2\to\mathrm{CWS}\Big[0,\ 0,\ \frac{5\,\widehat{tuv}}{2}-\frac{5\,\widehat{tvu}}{2}\Big]$

$3\to\mathrm{CWS}\Big[0,\ 3\,\widehat{tu}-4\,\widehat{uv},\ -\widehat{ttu}+2\,\widehat{tuu}-4\,\widehat{tuv}+\frac{13\,\widehat{tvu}}{2}-3\,\widehat{uuv}-\widehat{uvv}\Big]$

$4\to\mathrm{CWS}\big[0,\ -4\,\widehat{tu}+2\,\widehat{uv},\ -2\,\widehat{tuu}-3\,\widehat{tuv}-\widehat{tvu}+\widehat{uuv}+2\,\widehat{uvv}\big]$

$5\to\mathrm{CWS}\Big[0,\ -\widehat{tu}-2\,\widehat{uv},\ -\widehat{ttu}-\frac{\widehat{tuv}}{2}-\widehat{tvu}-2\,\widehat{uuv}+\widehat{uvv}\Big]$

6 → True

13.

14.

15. The growth map Γ

```
Γ[-1, ___] = MakeLieSeries[0];
Γ[n_, u_LW, γ_LieSeries, ss_] := Γ[n, u, γ, ss] = Module[{s, β0},
    β0 = Γ[n - 1, u, γ, s];
    ∫_0^ss (γ // DerivationExp[ad_u[s γ]] // adSeries[ad/(ⅇ^ad - 1), β0]) ⅆs
  ];
Γ[u_LW, γ_LieSeries, s_] := Γ[u, γ, s] = Module[{ser},
    ser = Unique[Γ];
    ser[] = Hold[Γ[u, γ]];
    ser[d_Integer] := ser[d] = Γ[d - 1, u, γ, s][d];
    LieSeries[ser]
  ];
Γ[u_LW, γ_LieSeries] := Γ[u, γ] = Module[{ser, s},
    ser = Unique[Γ];
    ser[] = Hold[γ[u, γ]];
    ser[d_Integer] := ser[d] = Γ[d - 1, u, γ, s][d] /. s → 1;
    LieSeries[ser]
  ];

Print /@ {
  0 → γ,
  1 → (t1 = Γ[u, γ]),
  2 → (t2 = β // DerivationExp[ad_u[γ]]),
  3 → (t3 = β // CC_u[Γ[u, γ]]),
  4 → t2 ≡ t3
 };
```

$$0 \to \mathrm{LS}\Big[-2\,\overline{t}-2\,\overline{u}+2\,\overline{v},\ \frac{\overline{tu}}{2}-\frac{3\,\overline{tv}}{2}-2\,\overline{uv},$$
$$-\frac{11}{6}\overline{t\,\overline{tu}}-\frac{1}{6}\overline{t\,\overline{tv}}-\frac{3}{2}\overline{t\,\overline{uv}}+\frac{1}{6}\overline{u\,\overline{uv}}-\frac{5}{3}\overline{\overline{tu}\,u}+\frac{11}{6}\overline{\overline{tv}\,u}+\frac{1}{3}\overline{\overline{tv}\,v}-\overline{\overline{uv}\,v}\Big]$$

$$1 \to \mathrm{LS}\Big[-2\,\overline{t}-2\,\overline{u}+2\,\overline{v},\ \frac{5\,\overline{tu}}{2}-\frac{3\,\overline{tv}}{2},$$
$$-3\,\overline{t\,\overline{tu}}-\frac{1}{6}\overline{t\,\overline{tv}}-\frac{4}{3}\overline{t\,\overline{uv}}-\frac{7}{6}\overline{u\,\overline{uv}}-\frac{17}{6}\overline{\overline{tu}\,u}+\frac{14}{3}\overline{\overline{tv}\,u}+\frac{1}{3}\overline{\overline{tv}\,v}+\frac{1}{3}\overline{\overline{uv}\,v}\Big]$$

$$2 \to \mathrm{LS}\Big[-\overline{t}+2\,\overline{u}+\overline{v},\ -4\,\overline{tu}-\frac{\overline{tv}}{2}-\frac{9\,\overline{uv}}{2},$$
$$\frac{19}{6}\overline{t\,\overline{tu}}+\frac{11}{6}\overline{t\,\overline{tv}}+\frac{61}{6}\overline{t\,\overline{uv}}+\frac{16}{3}\overline{u\,\overline{uv}}+\frac{1}{3}\overline{\overline{tu}\,u}+\frac{1}{6}\overline{\overline{tv}\,u}-\frac{5}{3}\overline{\overline{tv}\,v}+6\,\overline{\overline{uv}\,v}\Big]$$

$$3 \to \mathrm{LS}\Big[-\overline{t}+2\,\overline{u}+\overline{v},\ -4\,\overline{tu}-\frac{\overline{tv}}{2}-\frac{9\,\overline{uv}}{2},$$
$$\frac{19}{6}\overline{t\,\overline{tu}}+\frac{11}{6}\overline{t\,\overline{tv}}+\frac{61}{6}\overline{t\,\overline{uv}}+\frac{16}{3}\overline{u\,\overline{uv}}+\frac{1}{3}\overline{\overline{tu}\,u}+\frac{1}{6}\overline{\overline{tv}\,u}-\frac{5}{3}\overline{\overline{tv}\,v}+6\,\overline{\overline{uv}\,v}\Big]$$

4 → True

16. The many-variable growth map Γ

```
Γ[-1, ___] = MakeLieSeries[0];
Γ[n_, u_LW, us_List, γs : {_LieSeries ...}, ss_] :=
  Γ[n, u, us, γs, ss] = Module[{s, γ, Γ0},
    γ = u /. Thread[us → γs];
    Γ0 = Γ[n - 1, u, us, γs, s];
    ∫_0^ss (γ // DerivationExp[Sum[ad_{us[[i]]}[s γs[[i]]], {i, 1, Length[us]}]] // adSeries[ad/(e^ad - 1), Γ0]) ds
   ];
Γ[u_LW, us_List, γs : {_LieSeries ...}, s_] := Γ[u, us, γs, s] = Module[{ser},
    ser = Unique[Γ];
    ser[] = Hold[Γ[u, us, γs]];
    ser[d_Integer] := ser[d] = Γ[d - 1, u, us, γs, s][d];
    LieSeries[ser]
   ];
Γ[u_LW, us_List, γs : {_LieSeries ...}] := Γ[u, us, γs] = Module[{ser, s},
    ser = Unique[Γ];
    ser[] = Hold[γ[u, us, γs]];
    ser[d_Integer] := ser[d] = Γ[d - 1, u, us, γs, s][d] /. s → 1;
    LieSeries[ser]
   ];

Print /@ {
   0 → {α, β, γ},
   1 → (t1 = Γ[u, {u, v}, {α, β}]),
   2 → (t2 = β // DerivationExp[ad_u[α] + ad_v[β]]),
   3 → (t3 = β // CC_{u,v}[{Γ[u, {u, v}, {α, β}], Γ[v, {u, v}, {α, β}]}]),
   4 → t2 ≡ t3
  };
```

$$0 \to \Big\{\mathrm{LS}\Big[-2\,\overline{t}+\overline{u}-2\,\overline{v},\ -\frac{\overline{tu}}{2}+\overline{tv}+\frac{3\,\overline{uv}}{2},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}-2\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}+\overline{\overline{tu}\,u}-\frac{1}{3}\,\overline{\overline{tv}\,u}-\overline{\overline{tv}\,v}+\frac{1}{6}\,\overline{\overline{uv}\,v}\Big],\ \mathrm{LS}\Big[-\overline{t},\ -\overline{tu}+\frac{3\,\overline{tv}}{2}+\frac{\overline{uv}}{2},\ \frac{5}{3}\,\overline{t\,\overline{tu}}-\frac{1}{6}\,\overline{t\,\overline{tv}}-\frac{1}{2}\,\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+\frac{5}{3}\,\overline{\overline{tu}\,u}+\frac{5}{6}\,\overline{\overline{tv}\,u}+\frac{1}{6}\,\overline{\overline{tv}\,v}+\frac{5}{6}\,\overline{\overline{uv}\,v}\Big],\ \mathrm{LS}\Big[-\overline{t}+2\,\overline{u}+\overline{v},\ \overline{tu}+\frac{\overline{tv}}{2}+2\,\overline{uv},\ -\frac{1}{3}\,\overline{t\,\overline{tu}}+\frac{1}{3}\,\overline{t\,\overline{tv}}+\frac{7}{6}\,\overline{t\,\overline{uv}}-\frac{4}{3}\,\overline{u\,\overline{uv}}+\frac{7}{6}\,\overline{\overline{uv}\,v}\Big]\Big\}$$

$$1 \to \mathrm{LS}\Big[-2\,\overline{t}+\overline{u}-2\,\overline{v},\ -\frac{3\,\overline{tu}}{2}+2\,\overline{tv}+\frac{5\,\overline{uv}}{2},\ -\frac{5}{6}\,\overline{t\,\overline{tu}}-\frac{5}{2}\,\overline{t\,\overline{tv}}-\frac{11}{6}\,\overline{t\,\overline{uv}}-\frac{7}{12}\,\overline{u\,\overline{uv}}+\frac{7}{12}\,\overline{\overline{tu}\,u}+\frac{7}{12}\,\overline{\overline{tv}\,u}-\frac{17}{6}\,\overline{\overline{tv}\,v}+\frac{3}{2}\,\overline{\overline{uv}\,v}\Big]$$

$$2 \to \mathrm{LS}\Big[-\overline{t},\ -\overline{tu}+\frac{3\,\overline{tv}}{2}+\frac{\overline{uv}}{2},\ \frac{11}{3}\,\overline{t\,\overline{tu}}-\frac{5}{3}\,\overline{t\,\overline{tv}}-\frac{7}{2}\,\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+\frac{5}{3}\,\overline{\overline{tu}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,u}+\frac{1}{6}\,\overline{\overline{tv}\,v}+\frac{11}{6}\,\overline{\overline{uv}\,v}\Big]$$

$$3 \to \mathrm{LS}\Big[-\overline{t},\ -\overline{tu}+\frac{3\,\overline{tv}}{2}+\frac{\overline{uv}}{2},\ \frac{11}{3}\,\overline{t\,\overline{tu}}-\frac{5}{3}\,\overline{t\,\overline{tv}}-\frac{7}{2}\,\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+\frac{5}{3}\,\overline{\overline{tu}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,u}+\frac{1}{6}\,\overline{\overline{tv}\,v}+\frac{11}{6}\,\overline{\overline{uv}\,v}\Big]$$

4 → True

17.

18.

19.

20. The differential of BCH

```
Print /@ {
   1 → (bch = BCH[u, v]),
   2 → (BCH[u + ϵ t, v + ϵ w] - bch) / ϵ,
   3 → (t1 = (BCH[u + ϵ t, v + ϵ w] - bch) / ϵ // adSeries[(1 - ⅇ^(-ad)) / ad, bch]),
   4 → (t2 = t // adSeries[(1 - ⅇ^(-ad)) / ad, u] // Ad[-v]),
   5 → (t3 = w // adSeries[(1 - ⅇ^(-ad)) / ad, v])
  };
t1 ≡ t2 + t3
```

$1 \to \mathrm{LS}\left[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ \frac{1}{12}\,\overline{u\,\overline{uv}}+\frac{1}{12}\,\overline{\overline{uv}\,v}\right]$

$2 \to \mathrm{LS}\left[\overline{t}+\overline{w},\ \frac{\overline{tv}}{2}+\frac{\overline{uw}}{2},\ \frac{1}{12}\,\overline{t\,\overline{uv}}+\frac{1}{12}\,\overline{u\,\overline{uw}}+\frac{1}{12}\,\overline{u\,\overline{vw}}-\frac{1}{12}\,\overline{\overline{tv}\,u}+\frac{1}{12}\,\overline{\overline{tv}\,v}+\frac{1}{6}\,\overline{\overline{uw}\,v}\right]$

$3 \to \mathrm{LS}\left[\overline{t}+\overline{w},\ \frac{\overline{tu}}{2}+\overline{tv}-\frac{\overline{vw}}{2},\ \frac{1}{2}\,\overline{t\,\overline{uv}}+\frac{1}{6}\,\overline{v\,\overline{vw}}+\frac{1}{6}\,\overline{\overline{tu}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,v}\right]$

$4 \to \mathrm{LS}\left[\overline{t},\ \frac{\overline{tu}}{2}+\overline{tv},\ \frac{1}{2}\,\overline{t\,\overline{uv}}+\frac{1}{6}\,\overline{\overline{tu}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,v}\right]$

$5 \to \mathrm{LS}\left[\overline{w},\ -\frac{\overline{vw}}{2},\ \frac{1}{6}\,\overline{v\,\overline{vw}}\right]$

True

21. The differential of C

```
Print /@ {
   0 → {α, δα, γ},
   1 → (t1 = ((γ // CC[u, α + ϵ δα]) - (γ // CC[u, α])) / ϵ),
   2 → (t2 = γ // ad[u, δα // adSeries[(ⅇ^ad - 1)/ad, α] // RC[u, -α]] // CC[u, α]),
   t1 ≡ t2
  };
```

$0 \to \Big\{\mathrm{LS}\Big[\overline{t}-2\,\overline{u}+2\,\overline{v},\ -\overline{tv}+2\,\overline{uv},$
$-\frac{1}{2}\,\overline{t\,\overline{tu}}-\frac{11}{6}\,\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{1}{2}\,\overline{\overline{uv}\,v}\Big],$

$\mathrm{LS}\Big[\overline{v},\ -\frac{3\,\overline{tu}}{2}-2\,\overline{uv},\ -\frac{1}{3}\,\overline{t\,\overline{tu}}-\frac{5}{3}\,\overline{t\,\overline{tv}}-\frac{5}{3}\,\overline{t\,\overline{uv}}+\frac{11}{6}\,\overline{u\,\overline{uv}}-\frac{1}{2}\,\overline{\overline{tu}\,u}-2\,\overline{\overline{tv}\,u}+\frac{1}{3}\,\overline{\overline{tv}\,v}+\frac{2}{3}\,\overline{\overline{uv}\,v}\Big],$

$\mathrm{LS}\Big[-2\,\overline{t}-2\,\overline{u},\ -2\,\overline{tu}-2\,\overline{tv},$
$\frac{2}{3}\,\overline{t\,\overline{tu}}-\frac{1}{2}\,\overline{t\,\overline{tv}}+\frac{2}{3}\,\overline{t\,\overline{uv}}+\frac{11}{6}\,\overline{u\,\overline{uv}}-\frac{1}{2}\,\overline{\overline{tu}\,u}+\frac{7}{6}\,\overline{\overline{tv}\,u}-\frac{1}{6}\,\overline{\overline{tv}\,v}+\overline{\overline{uv}\,v}\Big]\Big\}$

$1 \to \mathrm{LS}\left[0,\ 2\,\overline{uv},\ 4\,\overline{t\,\overline{uv}}-6\,\overline{u\,\overline{uv}}+3\,\overline{\overline{tu}\,u}+\overline{\overline{tv}\,u}-4\,\overline{\overline{uv}\,v}\right]$

$2 \to \mathrm{LS}\left[0,\ 2\,\overline{uv},\ 4\,\overline{t\,\overline{uv}}-6\,\overline{u\,\overline{uv}}+3\,\overline{\overline{tu}\,u}+\overline{\overline{tv}\,u}-4\,\overline{\overline{uv}\,v}\right]$

True

22. The differential of $C_{u,v}$

```
Print /@ {
   0 → {α, δα, β, δβ, γ},
   1 → (t1 = 1/ϵ ((γ // CC_{u,v}[{α + ϵ δα, β + ϵ δβ}]) - (γ // CC_{u,v}[{α, β}]) )),
   2 → (t2 = Plus[
        γ // ad_u[δα // adSeries[(ⅇ^ad - 1)/ad, α] // RC_{u,v}[{-α, -β}]],
        γ // ad_v[δβ // adSeries[(ⅇ^ad - 1)/ad, β] // RC_{u,v}[{-α, -β}]]
      ] // CC_{u,v}[{α, β}]),
   t1 ≡ t2
  };
```

0 → {LS[$\overline{t}-2\,\overline{u}+2\,\overline{v}$, $-\overline{tv}+2\,\overline{uv}$,
$-\frac{1}{2}\overline{t\,\overline{tu}}-\frac{11}{6}\overline{t\,\overline{tv}}+\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-\frac{7}{6}\overline{\overline{tv}\,u}+\frac{1}{3}\overline{\overline{tv}\,v}+\frac{1}{2}\overline{\overline{uv}\,v}$],
LS[$\overline{v}$, $-\frac{3\,\overline{tu}}{2}-2\,\overline{uv}$, $-\frac{1}{3}\overline{t\,\overline{tu}}-\frac{5}{3}\overline{t\,\overline{tv}}-\frac{5}{3}\overline{t\,\overline{uv}}+\frac{11}{6}\overline{u\,\overline{uv}}-\frac{1}{2}\overline{\overline{tu}\,u}-2\,\overline{\overline{tv}\,u}+\frac{1}{3}\overline{\overline{tv}\,v}+\frac{2}{3}\overline{\overline{uv}\,v}$],
LS[$2\,\overline{t}-\overline{u}-\overline{v}$, $-\overline{tu}+\frac{\overline{tv}}{2}-\frac{\overline{uv}}{2}$, $\frac{1}{3}\overline{t\,\overline{tu}}-\frac{3}{2}\overline{t\,\overline{tv}}+\frac{1}{2}\overline{t\,\overline{uv}}+\frac{2}{3}\overline{u\,\overline{uv}}+\frac{3}{2}\overline{\overline{tu}\,u}+2\,\overline{\overline{tv}\,u}+\frac{11}{6}\overline{\overline{tv}\,v}$],
LS[$-2\,\overline{t}+2\,\overline{u}-2\,\overline{v}$, $-\overline{tu}-\frac{3\,\overline{tv}}{2}+2\,\overline{uv}$,
$\frac{5}{3}\overline{t\,\overline{tu}}-\frac{5}{3}\overline{t\,\overline{tv}}-\frac{5}{3}\overline{t\,\overline{uv}}-\frac{1}{3}\overline{u\,\overline{uv}}+\frac{5}{6}\overline{\overline{tu}\,u}-\frac{5}{6}\overline{\overline{tv}\,u}+2\,\overline{\overline{tv}\,v}-\overline{\overline{uv}\,v}$], LS[$-2\,\overline{t}-2\,\overline{u}$,
$-2\,\overline{tu}-2\,\overline{tv}$, $\frac{2}{3}\overline{t\,\overline{tu}}-\frac{1}{2}\overline{t\,\overline{tv}}+\frac{2}{3}\overline{t\,\overline{uv}}+\frac{11}{6}\overline{u\,\overline{uv}}-\frac{1}{2}\overline{\overline{tu}\,u}+\frac{7}{6}\overline{\overline{tv}\,u}-\frac{1}{6}\overline{\overline{tv}\,v}+\overline{\overline{uv}\,v}$]}

1 → LS[0, $2\,\overline{uv}$, $4\,\overline{t\,\overline{tv}}-6\,\overline{u\,\overline{uv}}+3\,\overline{\overline{tu}\,u}+\overline{\overline{tv}\,u}-4\,\overline{\overline{uv}\,v}$]

2 → LS[0, $2\,\overline{uv}$, $4\,\overline{t\,\overline{tv}}-6\,\overline{u\,\overline{uv}}+3\,\overline{\overline{tu}\,u}+\overline{\overline{tv}\,u}-4\,\overline{\overline{uv}\,v}$]

True

23. The differential of RC

```
Print /@ {
   0 → {α, β, γ},
   1 → (t1 = ((γ // RC[u, α + ϵ β]) - (γ // RC[u, α]))/ϵ),
   2 → (t2 = γ // RC[u, α] // ad[u, adSeries[(1 - ⅇ^-ad)/ad, α][β] // RC[u, α]]),
   t1 ≡ t2
  };
```

$0 \to \Big\{\mathrm{LS}\Big[\overline{t} - 2\,\overline{u} + 2\,\overline{v},\ -\overline{tv} + 2\,\overline{uv},$
$\quad -\frac{1}{2}\,\overline{t\,\overline{tu}} - \frac{11}{6}\,\overline{t\,\overline{tv}} + \overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - \frac{7}{6}\,\overline{\overline{tv}\,u} + \frac{1}{3}\,\overline{\overline{tv}\,v} + \frac{1}{2}\,\overline{\overline{uv}\,v}\Big],$
$\quad \mathrm{LS}\Big[2\,\overline{t} - \overline{u} - \overline{v},\ -\overline{tu} + \frac{\overline{tv}}{2} - \frac{\overline{uv}}{2},\ \frac{1}{3}\,\overline{t\,\overline{tu}} - \frac{3}{2}\,\overline{t\,\overline{tv}} + \frac{1}{2}\,\overline{t\,\overline{uv}} + \frac{2}{3}\,\overline{u\,\overline{uv}} + \frac{3}{2}\,\overline{\overline{tu}\,u} + 2\,\overline{\overline{tv}\,u} + \frac{11}{6}\,\overline{\overline{tv}\,v}\Big],$
$\quad \mathrm{LS}\Big[-2\,\overline{t} - 2\,\overline{u},\ -2\,\overline{tu} - 2\,\overline{tv},$
$\quad \frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{1}{2}\,\overline{t\,\overline{tv}} + \frac{2}{3}\,\overline{t\,\overline{uv}} + \frac{11}{6}\,\overline{u\,\overline{uv}} - \frac{1}{2}\,\overline{\overline{tu}\,u} + \frac{7}{6}\,\overline{\overline{tv}\,u} - \frac{1}{6}\,\overline{\overline{tv}\,v} + \overline{\overline{uv}\,v}\Big]\Big\}$

$1 \to \mathrm{LS}\Big[0,\ -4\,\overline{tu} - 2\,\overline{uv},\ -8\,\overline{t\,\overline{tu}} + 4\,\overline{t\,\overline{uv}} - \overline{u\,\overline{uv}} + 7\,\overline{\overline{tu}\,u} + 2\,\overline{\overline{tv}\,u} + 4\,\overline{\overline{uv}\,v}\Big]$

$2 \to \mathrm{LS}\Big[0,\ -4\,\overline{tu} - 2\,\overline{uv},\ -8\,\overline{t\,\overline{tu}} + 4\,\overline{t\,\overline{uv}} - \overline{u\,\overline{uv}} + 7\,\overline{\overline{tu}\,u} + 2\,\overline{\overline{tv}\,u} + 4\,\overline{\overline{uv}\,v}\Big]$

True

24. The differential of J

```
Print /@ {
   0 → {α, β};
   1 → (t0 = (J[u, α + ϵ β] - J[u, α]) / ϵ),
   2 → (t1 = div[u, β // adSeries[(1 - e^-ad) / ad, α] // RC[u, α]] // CC[u, -α]),
   t0 ≡ t1
  };
```

$1 \to \mathrm{CWS}\Big[-\widehat{u},\ -\frac{7\,\widehat{tu}}{2} + \frac{\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3} - \frac{9\,\widehat{tuu}}{2} - \frac{13\,\widehat{tuv}}{3} - \frac{19\,\widehat{tvu}}{12} - \frac{11\,\widehat{uuv}}{6} + \frac{13\,\widehat{uvv}}{6}\Big]$

$2 \to \mathrm{CWS}\Big[-\widehat{u},\ -\frac{7\,\widehat{tu}}{2} + \frac{\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3} - \frac{9\,\widehat{tuu}}{2} - \frac{13\,\widehat{tuv}}{3} - \frac{19\,\widehat{tvu}}{12} - \frac{11\,\widehat{uuv}}{6} + \frac{13\,\widehat{uvv}}{6}\Big]$

True