PROOF OF A CONJECTURE OF KULAKOVA ET AL. RELATED TO THE $\mathfrak{sl}_2$ WEIGHT SYSTEM

DROR BAR-NATAN AND HUAN T. VO

Abstract. In this article, we show that a conjecture raised in [KLMR], which regards the coefficient of the highest term when we evaluate the $\mathfrak{sl}_2$ weight system on the projection of a diagram to primitive elements, is equivalent to the Melvin-Morton-Rozansky conjecture, proved in [BNG].

1. Introduction

In this section, we briefly recall a conjecture of [KLMR] together with the relevant terminologies. A more complete treatment can be found in [KLMR]. Given a chord diagram D with m chords, its ***labelled intersection graph*** $\Gamma(D)$ is the simple labelled graph whose vertices are the chords of D, labelled from 1 to m, and two vertices are connected by an edge if the two corresponding chords intersect.

Following [KLMR], by orienting the chords of D arbitrarily, we can turn $\Gamma(D)$ into an oriented graph as follows. Given two intersecting oriented chords a and b, the edge connecting a and b goes from a to b if the beginning of the chord b belongs to the arc of the outer circle of D which starts at the tail of a and goes in the positive (counter-clockwise) direction to the head of a (see Figure 1). We also have another description of the orientation. Given two intersecting oriented chords a and b, we look at the smaller arc of the outer circle of D that contains the tails of a and b. Then we orient the edge connecting a and b from a to b if we go from the tail of a to the tail of b along the smaller arc in the counter-clockwise direction. The reader should check that the two definitions of orientation are equivalent.

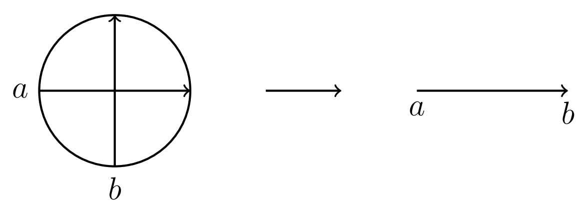

Figure 1. Orienting $\Gamma(D)$

Consider a circuit of even length $l = 2k$ in the oriented graph $\Gamma(D)$. By a ***circuit*** we mean a closed path in $\Gamma(D)$ with no repeated vertices. Choose an arbitrary orientation of the circuit. For each edge, we assign a weight $+1$ if the orientation of the edge coincides with the orientation of the circuit and -1 otherwise. The ***sign*** of a circuit is the product of the weights over all the edges in the circuit. We say that a circuit is ***positively oriented*** if its sign is positive and ***negatively oriented*** if its sign is negative. We define

$$R_k(D) := \sum_s \operatorname{sign}(s),$$

where the sum is over all (un-oriented) circuits s in $\Gamma(D)$ of length $2k$.

It is well-known that given a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant non-degenerate bilinear form, we can construct a weight system $w_{\mathfrak{g}}$ with values in the center $ZU((g))$ of the universal enveloping algebra $U(\mathfrak{g})$ (see, for instance [CDM, **Section 6**]). In the case of the Lie algebra $\mathfrak{sl}_2$, we obtain a weight system with values in the ring $\mathbb{C}[c]$ of polynomials in a single variable c, where c is the Casimir element of the Lie algebra $\mathfrak{sl}_2$. Note that the Casimir element c also depends on the choice of the bilinear form. For the case of $\mathfrak{sl}_2$, an ad-invariant non-degenerate bilinear form is given by

$$\langle x, y\rangle = \mathrm{Tr}(\rho(x)\rho(y)), \quad x, y \in \mathfrak{sl}_2,$$

where $\rho\colon \mathfrak{sl}_2 \to \mathfrak{gl}_2$ is the standard representation of $\mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ is simple, any invariant form is of the form $\lambda\langle\cdot,\cdot\rangle$ for some constant λ. If we let c_λ be the corresponding Casimir element and $c = c_1$, then $c_\lambda = c/\lambda$. If D is a chord diagram with n chords, it is known that

$$w_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + \cdots + a_1 c$$

and the weight system corresponding to $\lambda\langle\cdot,\cdot\rangle$ is

$$w_{\mathfrak{sl}_2,\lambda}(D) = c_\lambda^n + a_{n-1,\lambda}c_\lambda^{n-1} + \cdots + a_{1,\lambda}c_\lambda.$$

Therefore the relationship between these two weight systems is given by

$$w_{\mathfrak{sl}_2,\lambda}(D) = \frac{1}{\lambda^n}\, w_{\mathfrak{sl}_2}(D)\big|_{c=\lambda c_\lambda}.$$

Let $\mathcal{A}$ be the vector space generated by all chord diagrams (modulo the 1-term and 4-term relations). Then $\mathcal{A}$ is graded by the degree (number of chords) of a chord diagram, i.e.

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$$

where $\mathcal{A}_n$ is the space generated by all chord diagrams with n chords (modulo the 1-term and 4-term relations). Recall that we can turn $\mathcal{A}$ into a bialgebra by defining the following comultiplication

$$\Delta(D) = \sum_{V(D)=V_1\sqcup V_2} D\big|_{V_1} \otimes D\big|_{V_2},$$

where the sum is taken over all ordered disjoint partitions of $V(D)$, the set of chords of D. (Note that V_1 or V_2 can be empty.) An element $a \in \mathcal{A}$ is called ***primitive*** if

$$\Delta(a) = 1 \otimes a + a \otimes 1.$$

The set of all primitive elements forms a vector subspace of $\mathcal{A}$, which we denote by $\mathcal{P}$. On the other hand, a chord diagram D is called ***decomposable*** if it can be written as a product $D = D_1 \cdot D_2$ of two diagrams of smaller degrees. We let $\mathcal{D}$ denote the subspace spanned by the decomposable elements of $\mathcal{A}$. Note that $\mathcal{P}$ and $\mathcal{D}$ inherit a grading from $\mathcal{A}$ and $\mathcal{A}_n = \mathcal{P}_n \oplus \mathcal{D}_n$.

Following [L] we now define a map which projects $\mathcal{A}$ onto $\mathcal{P}$. Let D be a chord diagram with n chords, $V = V(D)$ its set of chords. Then the map π_n from the space of chord diagrams to its primitive elements is given by

$$\pi_n(D) = D - 1! \sum_{\{V_1,V_2\}} D|_{V_1} \cdot D|_{V_2} + 2! \sum_{\{V_1,V_2,V_3\}} D|_{V_1} \cdot D|_{V_2} \cdot D|_{V_3} - \cdots,$$

where sums are taken over all unordered disjoint partitions of V into non-empty subsets and $D|_{V_i}$ denotes D with only chords from V_i. If we change unordered partitions to ordered ones, we obtain

$$\pi_n(D) = D - \frac{1}{2}\sum_{V=V_1\sqcup V_2} D|_{V_1}\cdot D|_{V_2} + \frac{1}{3}\sum_{V=V_1\sqcup V_2\sqcup V_3} D|_{V_1}\cdot D|_{V_2}\cdot D|_{V_3} - \cdots . \tag{1}$$

It is shown (see [L]) that π_n is indeed a projection from $\mathcal{A}_n$ onto $\mathcal{P}_n$ along $\mathcal{D}_n$, i.e. it takes each primitive element into itself and it takes all decomposable elements to zero. Now we are ready to state the conjecture raised in [KLMR].

Conjecture 1. *Let D be a chord diagrams with $2m$ chords, and $w_{\mathfrak{sl}_2,2}$ be the weight system associated with $\mathfrak{sl}_2$ and $2\langle\cdot,\cdot\rangle$. Then*

$$w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = 2R_m(D)c_2^m + \text{ terms of degree less than } m \text{ in } c_2.$$

2. Proof of the conjecture

The conjecture is a consequence of the Melvin-Morton-Rozansky (MMR) conjecture, which was proved in [BNG]. We recall the statement of the MMR conjecture below. Let $J^k(q)$ be the "framing independent" colored Jones polynomial associated with the k-dimensional irreducible representation of $\mathfrak{sl}_2$. Set $q = e^h$, write $J^k(q)$ as power series in h:

$$J^k = \sum_{n=0}^{\infty} J^k_n h^n.$$

It is known that J^k_n is given by (see [O, **Theorem 6.14**] and [CDM, **Section 11.2.3**])

$$J^k_n = \mathrm{Tr}\left(w'_{\mathfrak{sl}_2}\big|_{c=\frac{k^2-1}{2}\cdot I_k}\right).$$

Here I_k is the $k\times k$ identity matrix and $w'_{\mathfrak{sl}_2}$ is the "deframing" of the weight system $w_{\mathfrak{sl}_2}$ (see [CDM, **Section 4.5.4**]). For any chord diagram D of degree n (modulo the framing independent relation), the value $w'_{\mathfrak{sl}_2}(D)$ is a polynomial in c of degree at most $\lfloor n/2\rfloor$ (see [CDM, **Exercise 6.25**]). It follows that J^k_n is a polynomial in k of degree at most $n+1$. Dividing J^k_n by k we then obtain

$$\frac{J^k}{k} = \sum_{n=0}^{\infty}\left(\sum_{0\le j\le n} b_{n,j}k^j\right)h^n,$$

where $b_{n,j}$ are Vassiliev invariants of order $\le n$. We denote the highest order part of the colored Jones polynomial by

$$JJ\colon = \sum_{n=0}^{\infty} b_{n,n}h^n.$$

Next we recall the definition of the Alexander-Conway polynomial of link diagrams. The Conway polynomial $C(t)$ can be defined by the skein relation:

(i) $C(\text{unknot}) = 1$,
(ii) $C(L_+) - C(L_-) = tC(L_0)$, where L_+, L_- and L_0 are identical outside the regions consisting of a positive crossing, a negative crossing and no crossing, respectively.

The Alexander-Conway polynomial is a Vassiliev power series:

$$\widetilde{C}(h) \colon = \frac{h}{e^{h/2} - e^{-h/2}} \, C|_{t=e^{h/2}-e^{-h/2}} = \sum_{n=0}^{\infty} c_n h^n.$$

Now we are ready to state the MMR conjecture, which was proved in [BNG].

Theorem. *With the notations as above, we have*

$$JJ(h)(K) \cdot \widetilde{C}(h)(K) = 1 \tag{2}$$

for any knot K.

The proof of the MMR conjecture found in [BNG] consists of reducing the equality of Vassiliev power series to an equality of weight systems. Recall that a Vassiliev invariant ν of order n gives us a weight system $W_n(\nu)$ of order n by $W_n(\nu)(D) = \nu(K_D)$, where D is a chord diagram of degree n and K_D is a singular knot whose chord diagram is D. Let

$$W_{JJ} \colon = \sum_{n=0}^{\infty} W_n(b_{n,n}) \text{ and } W_C \colon = \sum_{n=0}^{\infty} W_n(c_n).$$

Then it is shown in [BNG] that the equality (2) is equivalent to

$$W_{JJ} \cdot W_C = \mathbf{1}.$$

Here $\mathbf{1}$ denotes the weight system that takes value 1 on the empty chord diagram and 0 otherwise. Recall also that the product of two weight systems is given by

$$W_1 \cdot W_2(D) = m(W_1 \otimes W_2)(\Delta(D)),$$

where m denotes the usual multiplication in $\mathbb{Q}$. When D is primitive, we have

$$0 = W_{JJ} \cdot W_C(D) = m(W_{JJ} \otimes W_C)(D \otimes 1 + 1 \otimes D) = W_{JJ}(D) + W_C(D).$$

Thus we obtain

Lemma 1. *If D is a chord diagram of degree $2m$, then*

$$W_{JJ}(\pi_{2m}(D)) = -W_C(\pi_{2m}(D)).$$

To prove conjecture 1, we need the notion of ***logarithm*** of a weight system (see [LZ, **Chapter 6**]). Let w be a weight system and suppose w can be written as $w = \mathbf{1} + w_0$, where w_0 vanishes on chord diagrams of degree 0. Then

$$\log w \colon = \log(\mathbf{1} + w_0) = w_0 - \frac{1}{2} w_0^2 + \frac{1}{3} w_0^3 - \cdots$$

is well-defined since for each chord diagram we only have finitely many non-zero summands.

Lemma 2. *Let w be a multiplicative weight system, i.e. $w(D_1 \cdot D_2) = w(D_1)w(D_2)$, and $w($empty chord diagram$) = 1$. If D is a chord diagram of degree $2m$, then*

$$(\log w)(D) = w(\pi_{2m}(D)).$$

Proof. From the definition of the logarithm of a weight system we have

$$\begin{aligned}\log w &= \log(\mathbf{1} + (w - \mathbf{1})) \\ &= (w - \mathbf{1}) - \frac{1}{2}(w - \mathbf{1})^2 + \frac{1}{3}(w - \mathbf{1})^3 - \cdots\end{aligned}$$

Now if D is a chord diagram, then $(w-\mathbf{1})(\text{empty chord diagram}) = 0$ and $(w-\mathbf{1})(D) = w(D)$ if D has degree > 0. Therefore,

$$\begin{aligned}(w-1)^k(D) &= \sum_{V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k = V(D)} w(D|_{V_1}) w(D|_{V_2}) \cdots w(D|_{V_k}) \\ &= \sum_{V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k = V(D)} w(D|_{V_1} \cdot D|_{V_2} \cdots D|_{V_k}),\end{aligned}$$

where the sum is over ordered disjoint partition of $V(D)$ into non-empty subsets and the last equality follows from the multiplicativity of w. Comparing with equation (1) we obtain our desired equality. □

It is known that the weight system W_C is multiplicative. Therefore for a chord diagram D of degree $2m$,

$$(\log W_C)(D) = W_C(\pi_{2m}(D)).$$

Given an oriented circuit H in a labelled intersection graph, we define the ***descent*** $d(H)$ of the circuit to be the number of label-decreases of the vertices when we go along the circuit in the specified orientation. We have the following lemma.

Lemma 3. *Given a chord diagram D of degree $2m$, we have*

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D),$$

where the sum is over all oriented circuits H of length $2m$ in $\Gamma(D)$.

Proof. To prove the first equality, we show that by labelling the chords of D appropriately, the labelled intersection graph $\Gamma(D)$ of D has the property that the edges always go in the direction of increasing indices. To get a required labelling, we cut the outer circle of D to obtain a long chord diagram and then we label the chords by integers $1, 2, \ldots, 2m$ as we encounter them when we go from left to right in an increasing fashion. Then it's clear that a descent will correspond to an edge with weight -1. Every circuit H will have two possible orientations H_+ and H_-. However, since the circuit has even length, $d(H_+)$ and $d(H_-)$ have the same parity and the first equality follows.

The second equality is proved in [BNG, **Proposition 3.13**]. Here we briefly describe the main idea for completeness. The key identity is $W_C(D) = \det \mathrm{IM}(D)$, where $\mathrm{IM}(D)$ is the intersection matrix of the chord diagram D, which is defined as follows: label the chords of D as above, then $IM(D)$ is the $2m \times 2m$ matrix given by

$$\mathrm{IM}(D)_{ij} = \begin{cases} \mathrm{sign}(i-j) & \text{if chords } i \text{ and } j \text{ of } D \text{ intersect,} \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that $\mathrm{IM}(D)$ only depends on $\Gamma(D)$. The identity is proved by showing that $\det \mathrm{IM}$ satisfies the defining relations of W_C (see [BNG, **Theorem 3**]). Now expanding $\det \mathrm{IM}(D)$ we obtain

$$W_C(D) = \sum_{H=\bigcup_\alpha H_\alpha} (-1)^{\mathrm{sign}(\sigma_H)}(-1)^{d(H)}.$$

Here H is an oriented circuit of length $2m$ in $\Gamma(D)$, σ_H is the permutation of the vertices of $\Gamma(D)$ underlying H and $\bigcup_\alpha H_\alpha$ is the (unordered) cycle decomposition of σ_H. From there it follows that

$$(\log W_C)(D) = -\sum_H (-1)^{d(H)}.$$

The readers can consult [BNG] for more details. □

Proof of Conjecture 1. Let D be a chord diagram of degree $2m$, we have a chain of equalities from the above lemmas

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D) = -W_C(\pi_{2m}(D)) = W_{JJ}(\pi_{2m}(D)).$$

Therefore,

$$\frac{J^k_{2m}(\pi_{2m}(D)}{k} = 2R_m(D)k^{2m} + \cdots .$$

Plug in $c = (k^2-1)/2$ or $k^2 = 2c+1$ we obtain

$$w_{\mathfrak{sl}_2}(\pi_{2m}(D)) = 2^{m+1}R_m(D)c^m + \cdots .$$

Now we just need to do a change of variable

$$w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = \frac{1}{2^{2m}}\, w_{\mathfrak{sl}_2}(\pi_{2m}(D))|_{c=2c_2} = 2R_m(D)c_2^m + \cdots$$

and the proof is complete. □

Remark. Technically we need to consider $w'_{\mathfrak{sl}_2}$ instead of $w_{\mathfrak{sl}_2}$. However for primitive elements, deframing does not affect the value of the highest terms (see [CDM, **Section 4.5.4**]).

References

[BNG] **Dror Bar-Natan and Stavros Garoufalidis**, *On the Melvin-Morton-Rozansky Conjecture.* Inventiones Mathematicae, vol. **125** (1996) 103–133.

[CDM] **S. Chmutov, S. Duzhin and J. Mostovoy**, *Introduction to Vassiliev Knot Invariants.* Cambridge University Press, 2012.

[KLMR] **E. Kulakova, S. Lando, T. Mukhutdinova and G. Rybnikov**, *On a Weight System Conjecturally Related to $\mathfrak{sl}_2$.* 2013, arXiv:1307.4933.

[L] **S. K. Lando**, *On Primitive Elements in the Bialgebra of Chord Diagrams.* Amer. Math. Soc. Transl. Ser. 2, AMS, Providence RI, vol. **180** (1997) 167–174.

[LZ] **S. Lando and A. Zvonkin**, *Graphs on Surfaces and Their Applications.* Springer, 2004.

[O] **Tomotada Ohtsuki**, *Quantum Invariants: a Study of Knots, 3-Manifolds and Their Sets.* World Scientific, 2002.

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
E-mail address: drorbn@math.toronto.edu
URL: http://www.math.toronto.edu/~drorbn

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
E-mail address: vohuan@math.toronto.edu