Willwacher: Jointly Orthogonal Polynomials

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(joint work with G. Felder)

Classical theory

- $\langle\, , \rangle$ an inner product on $\mathbb{R}[x]$, typically
$$\langle f,g\rangle = \int_I f(x)g(x)w(x)dx$$

- Find a basis using Gram-Schmidt: $e_0, e_1, \ldots\ldots$ s.t.

  e_n is the unique $\neq 0$, up to scalar in $V_{n-1}^{\perp} \subset V_n \subset \mathbb{R}[x]$

  V_n: poly's of deg $\leq n$

- Often solutions of ODES

Jacobi	Laguerre	Hermite
$I=(-1,1)$	$(0,\infty)$	$\mathbb{R}$
$w=(1-x)^{\alpha}(1+x)^{\beta}$	$w=x^{\alpha}e^{-x}$	$e^{-x^2/2}$

$$Lf = (1-x^2)f'' + (\beta-\alpha-(\alpha+\beta+2)x)f'$$
$$Lf = \lambda f$$

$\rightsquigarrow$ sol'ns of these diff eq's are uniquely characterised by their orthogonality properties.

Many ODEs produce polynomial sol'ns that are orthogonal w.r.t. multiple inner products

Example

$$\langle f,g\rangle = \int_{I_j} fg\, w(x)\,dx$$

some collection of disjoint intervals.

Lamé diff eq:

$$Q(x)f'' + \tfrac{1}{2}Q'(x)f'(x) - \tfrac{1}{4}(\nu(\nu+1)x+\lambda)f = 0$$

$(x-e_1)(x-e_2)(x-e_3)$

$e_1 < e_2 < e_3$

If $\nu = 2n$ there are $n+1$ sol'ns that are poly of degree n.

$$\langle f,g\rangle_{1,2} = \int_{I_{1,2}} fg\,w\,dx \qquad I_1 = (e_1,e_2) \quad I_2 = (e_2,e_3)$$

$$w = \prod (x-e_i)^{-1/2}$$

other examples: * Ince polynomials $(-1,1), (1,\infty)$

* $I_1 = (-\infty, 0) \quad I_2 = (0,\infty)$

Occurs in the study of Schrödinger eq'ns with $O(r^6)$ potential.

Question Are such polynomials uniquely determined by such orthogonality properties.

Ans Yes.

Higher examples "Heine–Stieltjes polys"

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⋮

Joint orthogonality and $\overset{\text{eigenvalue}}{\text{EV}}$ problems.

$K=2$

Inner products $\rightsquigarrow$ $A_{1,2}$ symmetric matrices

simultaneously diagonalize $\Leftrightarrow$ solve relative eqn:

$$(\lambda_1 A_1 + \lambda_2 A_2) v = 0 \qquad (\lambda_1 \lambda_2) \in \mathbb{CP}^2/\mathbb{RP}^2$$

⋮