Gr(2,4)

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(131219) $Gr(\mathbb{R}^2 \hookrightarrow \mathbb{R}^4) = S^2 \times S^2$. Karshon: • There's a $\mathbb{C}P^2 = S^2$ of complex lines in $\mathbb{C} \times \mathbb{C}$ and in $\mathbb{C} \times \bar{\mathbb{C}}$. • It is the product of the moduli $C(\mathbb{R}^4) \times C(\bar{\mathbb{R}}^4)$ of metric complex structures on $\mathbb{R}^4$ / $\bar{\mathbb{R}}^4$. For $(I, \bar{I}) \in C(\mathbb{R}^4) \times C(\bar{\mathbb{R}}^4)$ there is a unique $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ which is complex relative to both, and a given $P = \mathbb{C} = \mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ determines two metric complex structures on $\mathbb{R}^4/\bar{\mathbb{R}}^4$ by multiplication by i on P and by $\pm i$ on $P^\perp$. Finally $C(\mathbb{R}^4) = SO(4)/U(2) =$ {left multiplications L_u by unit imaginary quaternions u} $= S^2$ and $C(\bar{\mathbb{R}}^4) = \{R_v\}_{v \in S^2 \subset \mathbb{R}^3 \subset \mathbb{H}}$. • $P(u,v) = \text{span}(u+v, uv-1)$ or $\text{span}(u-v, uv+1)^\perp$ and for orthonormal (α,β), $\text{span}(\alpha,\beta) \mapsto (\beta\bar{\alpha}, \bar{\alpha}\beta)$.

<u>Claim</u> If α, β are orthonormal and

$$(u,v) := (\beta\bar{\alpha}, \bar{\alpha}\beta),$$

then $\text{span}(\alpha,\beta) \perp \text{span}(u-v, uv+1)$

<u>Proof</u>

$$\langle \alpha, u-v \rangle = Re(\alpha(\alpha\bar{\beta} - \bar{\beta}\alpha)$$

$$= Re(\alpha(\bar{\alpha}\beta - \beta\bar{\alpha}) =$$

$$= Re(\beta - \alpha\beta\bar{\alpha}) = \tfrac{1}{2}(\beta - \alpha\beta\bar{\alpha} + \bar{\beta} - \alpha\bar{\beta}\bar{\alpha}) = \tfrac{1}{2}(\beta + \bar{\beta} - \alpha(\beta+\bar{\beta})\bar{\alpha}) = 0.$$

$$\langle \beta, u-v \rangle = Re(\bar{\beta}(\beta\bar{\alpha} - \bar{\alpha}\beta) =$$

$$= Re(\bar{\alpha} - \bar{\beta}\bar{\alpha}\beta) = \cdots = 0$$

$$\langle \alpha, uv+1 \rangle = Re(\alpha(\bar{\beta}\alpha\alpha\bar{\beta} + 1))$$

$$= \tfrac{1}{2}(\alpha\bar{\beta}\alpha\alpha\bar{\beta} + \alpha + \beta\bar{\alpha}\bar{\alpha}\beta\bar{\alpha} + \bar{\alpha})$$

$$\alpha, \beta \mapsto (u = \overset{= -\alpha\bar{\beta}}{\beta\bar{\alpha}}, v = \overset{= -\bar{\beta}\alpha}{\bar{\alpha}\beta}) \mapsto$$

$$\text{span}(\beta\bar{\alpha} + \bar{\alpha}\beta, \beta\bar{\alpha}\bar{\alpha}\beta - 1)$$

$$-u = \bar{u}, -v = \bar{v}, \quad u^2 = v^2 = -1$$

$$Re((u+v)\overline{(u-v)}) = Re(u\bar{u} - v\bar{v} + v\bar{u} - u\bar{v}) = 0$$

$$\alpha \perp \beta\bar{\alpha} - \bar{\alpha}\beta \ ? \qquad \overline{\beta\bar{\alpha}} = \alpha\bar{\beta}$$

$$Re(\alpha(\alpha\bar{\beta} - \bar{\beta}\alpha)) = Re(\alpha($$

$$Re(\alpha$$

$$\langle uv-1, u-v \rangle = Re((1-uv)\cdot(u-v)) =$$

$$Re(\cancel{u} - v - uvu + \cancel{uv^2})$$

$$= Re(-v - uvu)$$

$$= \tfrac{1}{2}(-v - uvu + v + uvu) = 0$$

$$Re(\alpha(\beta\bar{\alpha} - \bar{\alpha}\beta)) = \tfrac{1}{2}(\alpha\beta\bar{\alpha} - \alpha\bar{\alpha}\beta + \alpha\bar{\beta}\bar{\alpha} - \bar{\beta})$$

$$= \tfrac{1}{2}(\alpha(\beta+\bar{\beta})\bar{\alpha} - (\beta+\bar{\beta})) = 0.$$

Note

$$\bar{\alpha}\beta = -\bar{\beta}\alpha$$

$$\alpha\bar{\beta} = -\beta\bar{\alpha}$$

$$= \tfrac{1}{2}(\alpha\beta\alpha\alpha\beta + \alpha + \beta\alpha\alpha\beta\alpha + \alpha)$$

$$= \tfrac{1}{2}(-\alpha\bar{\alpha}\beta\alpha\bar{\beta} + \alpha - \beta\bar{\alpha}\bar{\beta}\alpha\bar{\alpha} + \bar{\alpha})$$

$$= \tfrac{1}{2}(-\beta\alpha\bar{\beta} - \beta\bar{\alpha}\bar{\beta} + \alpha + \bar{\alpha}) = 0.$$

Invariance under inner rotations:

$$\alpha \longmapsto c\alpha + s\beta$$
$$\beta \longmapsto -s\alpha + c\beta$$

$$\beta\bar{\alpha} \mapsto (-s\alpha + c\beta)(c\bar{\alpha} + s\bar{\beta}) = -\cancel{sc} - s^2\alpha\bar{\beta}$$
$$+ c^2\beta\bar{\alpha} + \cancel{sc} = (c^2 + s^2)\beta\bar{\alpha}$$

$$(u,v) \longmapsto (\underset{\alpha}{u+v},\ \underset{\beta}{uv-1}) \longmapsto \beta\bar{\alpha} = (uv-1)(\overline{u+v}) =$$

$$= (1-uv)(u+v) = u + v - uvu - uv^2$$

$$\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = \alpha \qquad \beta = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$$

$$\beta\bar{\alpha} = (\beta_0 + \beta_1 i + \beta_2 j + \beta_3 k)(\alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k) =$$

$$= (\beta_0\alpha_0 + \beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3)$$
$$+ (\beta_1\alpha_0 - \beta_0\alpha_1 - \beta_2\alpha_3 + \beta_3\alpha_2)\, i + \ldots$$

$$\bar{\alpha}\beta = (\alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k)(\beta_0 + \beta_1 i + \beta_2 j + \beta_3 k)$$
$$= (\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)$$
$$+ (\alpha_0\beta_1 - \alpha_1\beta_0 + \beta_2\alpha_3 - \beta_3\alpha_2)\, i + \ldots$$

$$\begin{pmatrix} \alpha_0\beta_0 & \alpha_0\beta_1 & \alpha_0\beta_2 & \alpha_0\beta_3 \\ \alpha_1\beta_0 & \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1\beta_0 & \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 \\ \alpha_2\beta_0 & \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 \\ \alpha_3\beta_0 & \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 \end{pmatrix}$$